An abelian quotient of the symplectic derivation Lie algebra of the free Lie algebra

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The symplectic derivation Lie algebra $\mathfrak{h}_{g,1}$

- $\Sigma_{g,1}$: a compact oriented connected surface of genus g
  w/ one boundary component

- $H := H_1(\Sigma_{g,1}; \mathbb{Q}) \cong \mathbb{Q}^{2g}$

- Intersection form on H:

$$\mu : H \otimes H \longrightarrow \mathbb{Q} \qquad \left(\begin{array}{l} \text{non-degenerate} \\ \text{skew-symmetric} \end{array} \right)$$

- $\mathrm{Sp}(H) \cong \mathrm{Sp}(2g, \mathbb{Q})$: symplectic group,

$$\mathrm{Sp}(H) \curvearrowright H \qquad \mu\text{-preserving action.}$$

- Let
$$\mathfrak{h}_{g,1} = \bigoplus_{k=0}^{\infty} \mathfrak{h}_{g,1}(k)$$
be the graded Lie algebra of the symplectic derivations of the free Lie algebra $\mathcal{L}(H) = \bigoplus_{i=1}^{\infty} \mathcal{L}_i(H)$ generated by H.
- $\mathfrak{h}_{g,1}(0) \cong \mathfrak{sp}(2g, \mathbb{Q}) \cong S^2 H$
- For $k \geq 1$, we have
$$\mathfrak{h}_{g,1}(k) = \mathrm{Ker}\left(\mathcal{L}_1(H) \otimes \mathcal{L}_{k+1}(H) \xrightarrow{[\cdot,\cdot]} \mathcal{L}_{k+2}(H)\right).$$
- $\mathfrak{h}_{g,1}^{+} := \bigoplus_{k=1}^{\infty} \mathfrak{h}_{g,1}(k)$: the Lie ideal of the positive degree part.

Problem.

Determine the homology group

$$H_*(\mathfrak{h}_{g,1}^+) = H_*(\wedge^*(\mathfrak{h}_{g,1}^+), \partial).$$

- The grading of $\mathfrak{h}_{g,1}^+$ induces

$$H_n(\mathfrak{h}_{g,1}^+) = \bigoplus_{w=0}^{\infty} H_n(\mathfrak{h}_{g,1}^+)_w,$$

where $H_n(\mathfrak{h}_{g,1}^+)_w$ is the weight w-part.

Theorem. [Kontsevich, 1993]

$$PH_k(\mathfrak{h}_{\infty,1}^+)_{2n}^{\mathrm{Sp}} \cong H^{2n-k}(\mathrm{Out}\ F_{n+1}; \mathbb{Q}) \quad n \geq 1,\ k \geq 1$$

Here $\mathfrak{h}_{\infty,1}^+ := \lim_{g\to\infty} \mathfrak{h}_{g,1}^+$.

- We computed the integral Euler characteristics

$$e(\mathrm{Out}\, F_n) = \sum_{i=0}^{2n-3} (-1)^i \dim \left(H^i(\mathrm{Out}\, F_n; \mathbb{Q})\right)$$

of $\mathrm{Out}\, F_n$ up to $n \leq 11$. Recall that $\mathrm{vcd}(\mathrm{Out}\ F_n) = 2n-3$.

Theorem. [MSS, 2015]

n	2	3	4	5	6	7	8	9	10	11
$e(\mathrm{Out}\, F_n)$	1	1	2	1	2	1	1	-21	-124	-1202

- It had been shown that

$H^4(\mathrm{Out}\, F_4; \mathbb{Q}) \cong \mathbb{Q}\langle \mu_1^* \rangle$ (Hatcher-Vogtmann, M. 1998),

$H^8(\mathrm{Out}\, F_6; \mathbb{Q}) \cong \mathbb{Q}\langle \mu_2^* \rangle$ (Ohashi, 2008),

$H^{12}(\mathrm{Out}\, F_8; \mathbb{Q}) \supset \mathbb{Q}\langle \mu_3^* \rangle$ (Gray, 2011).

- Our computation shows the existence of many non-trivial odd-dimensional classes in $H^*(\mathrm{Out}\, F_n; \mathbb{Q})$ for $8 \leq n \leq 11$.

Problem.

Determine $H_1(\mathfrak{h}^+_{g,1})^{\mathrm{Sp}}$ and $H_1(\mathfrak{h}^+_{\infty,1})^{\mathrm{Sp}}$

- The weight w part of $H_1(\mathfrak{h}^+_{g,1})$ is

$$H_1(\mathfrak{h}^+_{g,1})_w := \mathfrak{h}_{g,1}(w) \Big/ \sum_{i=1}^{w-1} [\mathfrak{h}_{g,1}(i), \mathfrak{h}_{g,1}(w-i)].$$

- Kontsevich's theorem says that

$$H_1\left(\mathfrak{h}^+_{\infty,1}\right)^{\mathrm{Sp}}_{2n} \cong H^{2n-1}(\mathrm{Out}\ F_{n+1}; \mathbb{Q})$$

for $n \geq 1$. Here $\mathrm{vcd}(\mathrm{Out}\ F_{n+1}) = 2n-1$.

- $H_1\left(\mathfrak{h}^+_{g,1}\right)^{\mathrm{Sp}}_{2k-1} = 0$ for $k \geq 1$
- $H_1\left(\mathfrak{h}^+_{g,1}\right)^{\mathrm{Sp}}_{2k}$ stabilizes for $g \gg k$.

- Recently, Bartholdi (2015) showed

$$H^p(\mathrm{Out}\ F_7;\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & (p = 0, 8, 11) \\ 0 & (\text{otherwise}) \end{cases}$$

with the aid of computers.
(Need to compute the rank of a 2038511×536647 matrix)

- $H^{11}(\mathrm{Out}\ F_7;\mathbb{Q}) \cong \mathbb{Q}$ is remarkable because it is the first non-trivial odd and (virtually) top rational cohomology group which is explicitly described.

Main result

- By theorems of Kontsevich and Bartholdi, we have

$$H_1(\mathfrak{h}_{\infty,1}^+)_{12}^{\mathrm{Sp}} \cong H^{11}(\mathrm{Out}\ F_7; \mathbb{Q}) \cong \mathbb{Q}.$$

- We proved $H_1(\mathfrak{h}_{\infty,1}^+)_{12}^{\mathrm{Sp}} = H_1(\lim_{g\to\infty} \mathfrak{h}_{g,1}^+)_{12}^{\mathrm{Sp}} \cong \mathbb{Q}$ directly in $\mathfrak{h}_{g,1}^+$.

- It gives an alternative proof of $H^{11}(\mathrm{Out}\, F_7; \mathbb{Q}) \cong \mathbb{Q}$.

- Our proof also uses computers.

More precisely,

Theorem 1. [MSS, 2016]

There exists an $\mathrm{Sp}(2g,\mathbb{Q})$-invariant linear map

$$C : \mathfrak{h}_{g,1}(12) \longrightarrow \mathbb{Q}$$

satisfying that

- C is non-trivial for any $g \geq 2$,
- the restriction of C to $\displaystyle\sum_{i=1}^{11}[\mathfrak{h}_{g,1}(i), \mathfrak{h}_{g,1}(12-i)]$ is trivial.

That is, the cocycle C gives a surjection

$$\widetilde{C} : H_1(\mathfrak{h}_{g,1}^{+})_{12}^{\mathrm{Sp}} \longrightarrow\!\!\!\!\rightarrow \mathbb{Q}$$

for every $g \geq 2$. Moreover $\widetilde{C}$ is an isomorphism for $g \geq 8$.

- Since $H_1(\mathfrak{h}_{1,1}^{+})_{12}^{\mathrm{Sp}} = 0$, our bound of genus for the non-triviality of $H_1(\mathfrak{h}_{g,1}^{+})_{12}^{\mathrm{Sp}}$ is best possible.

Method for computation of $H_1(\mathfrak{h}^+_{g,1})^{\mathrm{Sp}}_{12}$

1. Find a coordinate system of $\mathfrak{h}_{g,1}(12)^{\mathrm{Sp}} \cong \mathbb{Q}^{650}$.

   Actually, we did it in $\mathfrak{h}_{g,1}(12)^{\mathrm{Sp}} \subset \left(H^{\otimes 14}\right)^{\mathrm{Sp}} \cong \mathbb{Q}^{135135}$.

2. Compute the bracket map

$$[\,\cdot\,,\,\cdot\,] : \left(\bigoplus_{i=1}^{6} \left(\mathfrak{h}_{g,1}(i) \otimes \mathfrak{h}_{g,1}(12-i)\right)\right)^{\mathrm{Sp}} \longrightarrow \mathfrak{h}_{g,1}(12)^{\mathrm{Sp}}.$$

   We see that the image includes a subspace $W \cong \mathbb{Q}^{649}$.

3. Find a linear map $C : \mathfrak{h}_{g,1}(12)^{\mathrm{Sp}} \twoheadrightarrow \mathbb{Q}$ which annihilates W.

4. Check that C is trivial on the image of the bracket map.

We obtained C as a linear comb. of 647 multiple contractions.

Relationship with the Enomoto-Satoh map

Theorem 2. [MSS, 2016]

For $g \geq 6$, the $\mathrm{Sp}(2g, \mathbb{Q})$-invariant cocycle $C : \mathfrak{h}_{g,1}(12) \longrightarrow \mathbb{Q}$ factors through the Enomoto-Satoh map

$$ES_{12} : \mathfrak{h}_{g,1}(12) \hookrightarrow H \otimes \mathcal{L}_{13}(H) \hookrightarrow H^{\otimes 14} \xrightarrow{\mu \otimes (\mathrm{id}^{\otimes 12})} H^{\otimes 12} \longrightarrow \left(H^{\otimes 12}\right)_{\mathbb{Z}/12\mathbb{Z}}.$$

- This theorem provides another description of the map C in the form
$$C = C' \circ ES_{12}$$
with C' described by chord diagrams with 6 chords, which serve as coordinate functions of $\left(H^{\otimes 12}\right)^{\mathrm{Sp}}_{\mathbb{Z}/12\mathbb{Z}} \cong \mathbb{Q}^{897}$.
- We obtained C' as a linear comb. of 278 multiple contractions.

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Fin.