# Reductivity problem on knot projections

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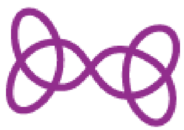
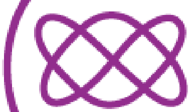
# §0. Outline

How reduced P is

P: a knot projection

$r(P)$: the **reductivity** of P

**Example**

$r($  $)=0$ $r($  $)=1$ $r($  $)=2$ $r($  $)=3$

**Theorem (S.)** $r(P) \leq 4 \quad ({}^{\forall}P)$

**Reductivity problem**

${}^{\exists ?}P$ s.t. $r(P) = 4$ → If exists, P should ...?

# Contents



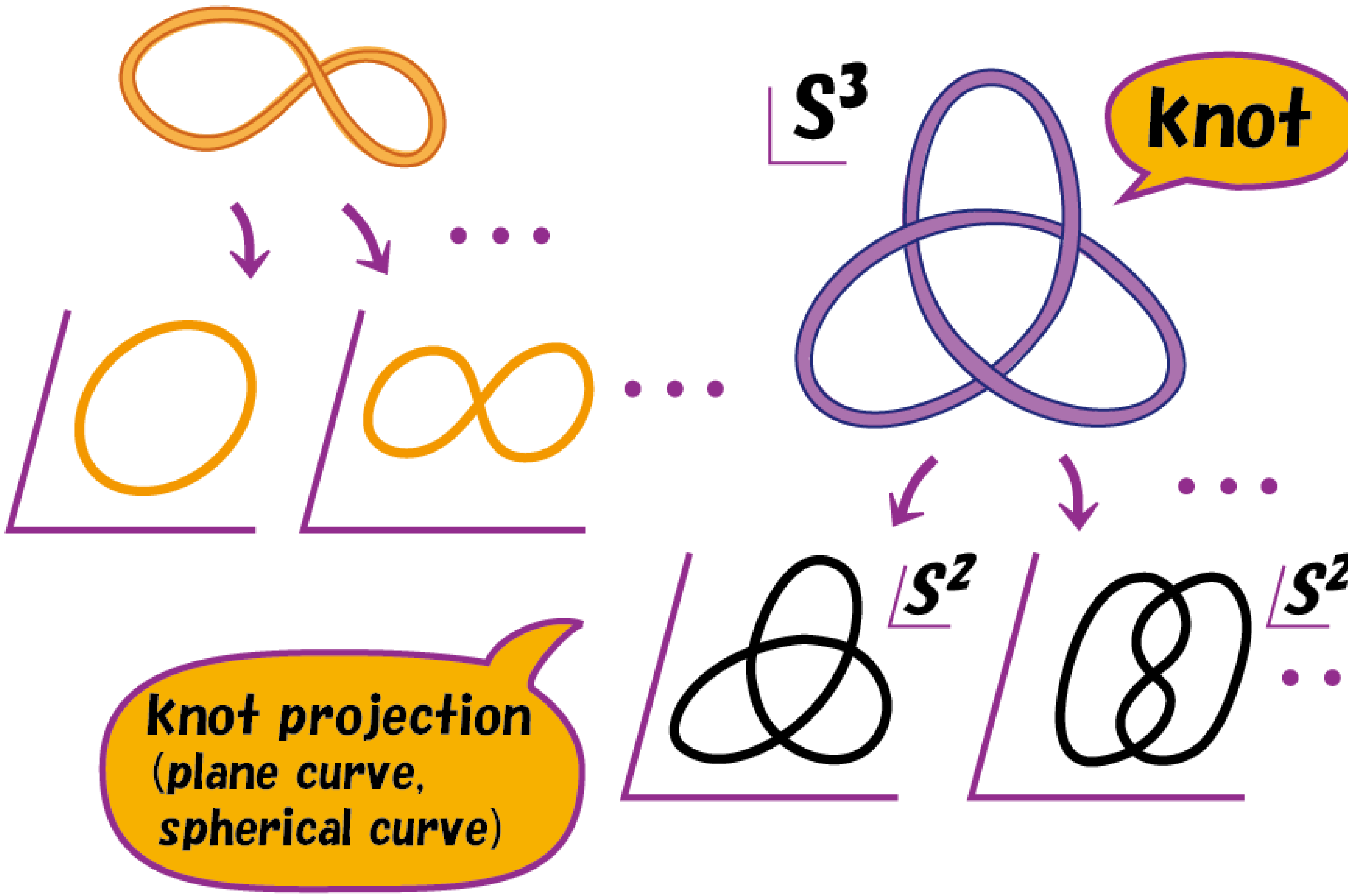
joint work with

**Yui Onoda**

arXiv:1603.07811

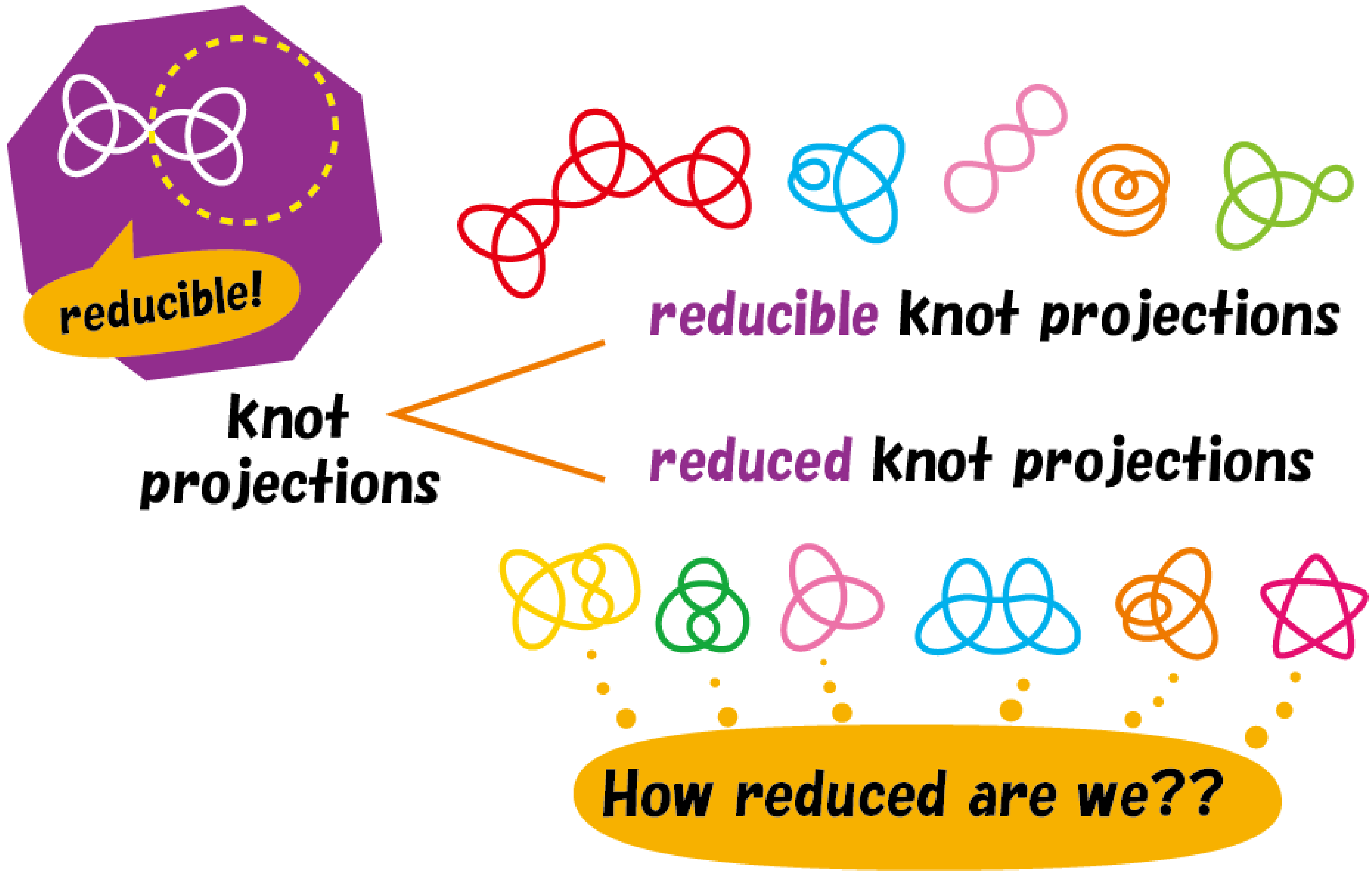
# §1. Knot projections

# Knot projection




We assume knot projections have at least one crossing.

# Reducible and reduced

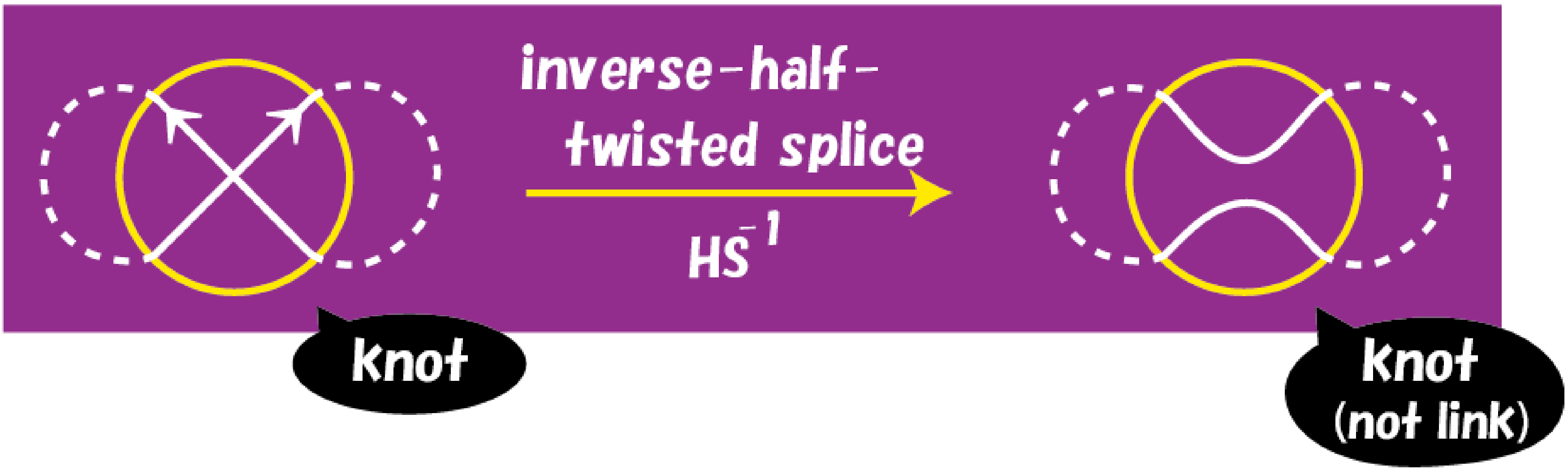


# §2. Reductivity

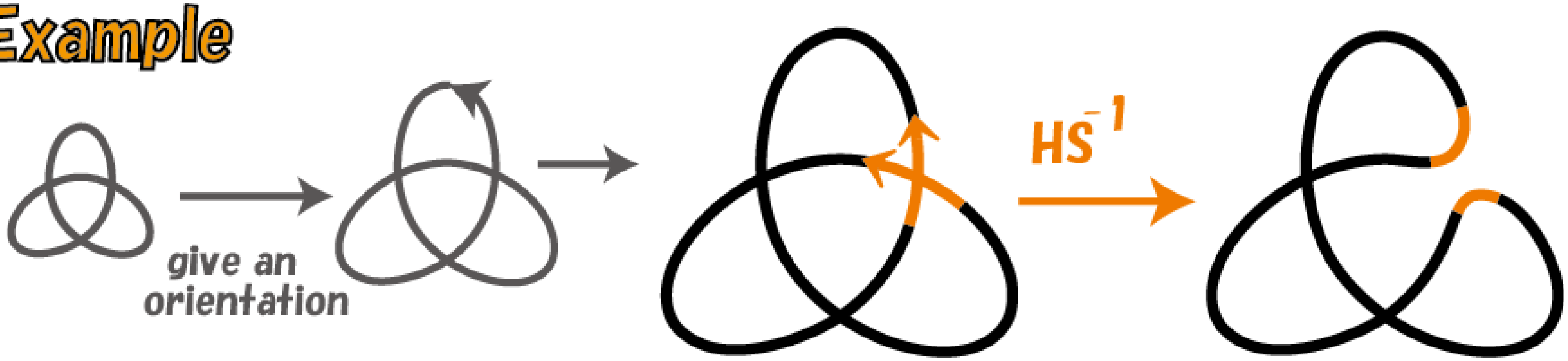
How reduced?

How far from reducible?

# Inverse-half-twisted splice ($HS^{-1}$)

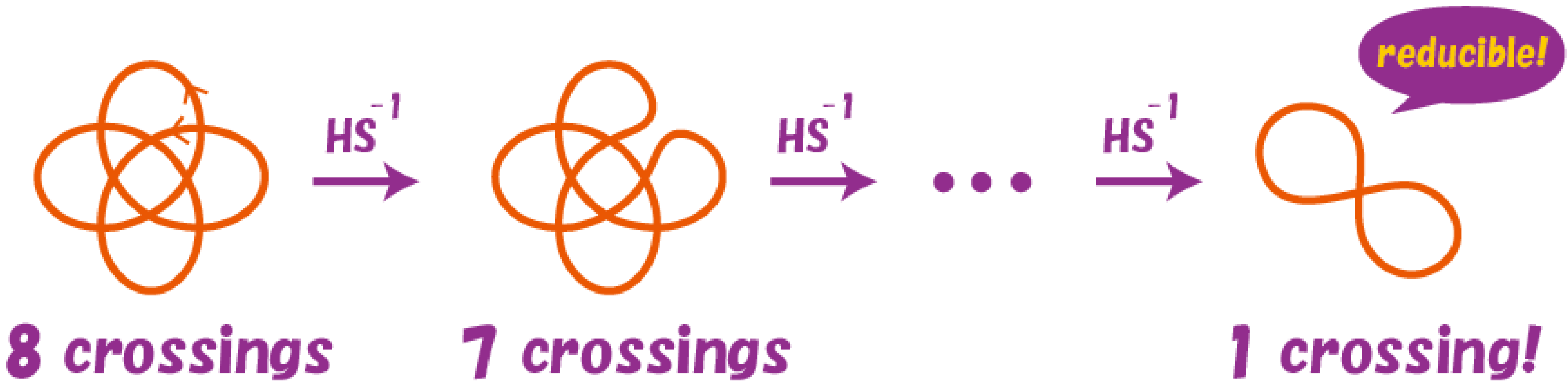



## Example



## Remark

We can obtain a reducible knot projection from any knot projection by a finite number of $HS^{-1}$!



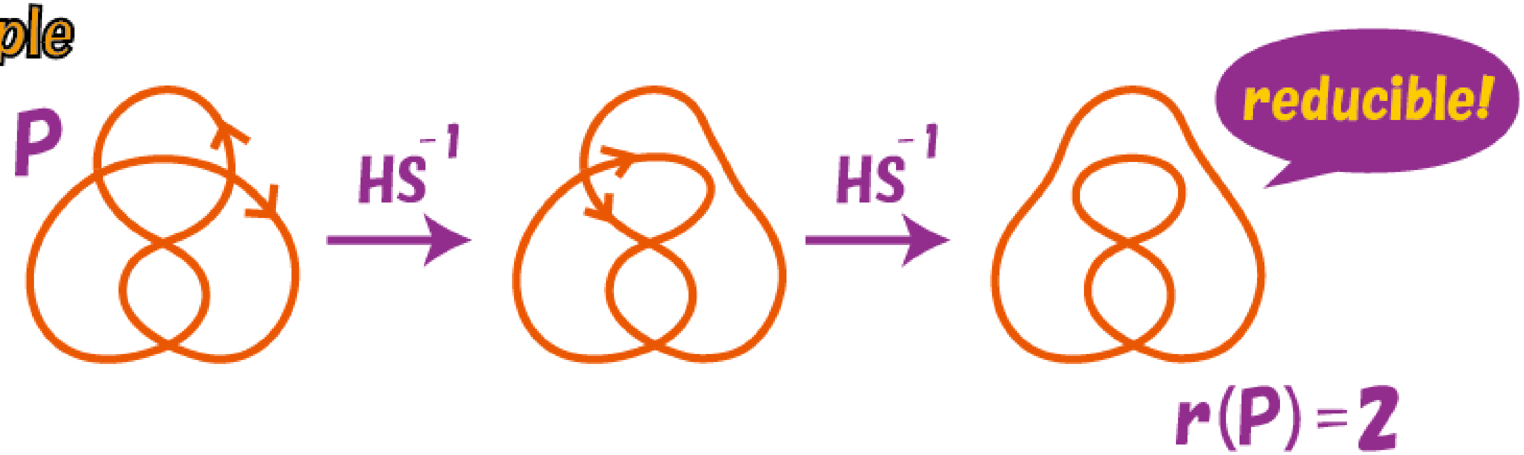
# Reductivity -how reduced??

P: a knot projection

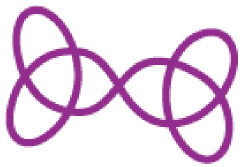
## Definition

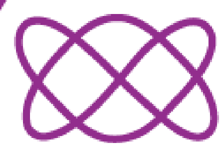
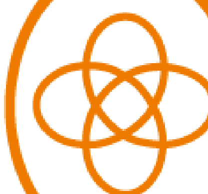
The **reductivity** r(P) of P is the minimal number of $HS^{-1}$ which are needed to obtain a reducible knot projection from P.

### Example



## Example

r(  ) = 0   r(  ) = 1

r(  ) = 2   r(  ) = 3

## Remark

There exist infinitely many knot projections P with r(P) = 0, 1, 2, and 3.

# Reductivity is four or less

**Theorem 1 (S)**

$$r(P) \leq 4 \quad ({}^{\forall}P)$$

**Reductivity problem**

$${}^{\exists ?}P \text{ s.t. } r(P) = 4$$

Reference: A. Shimizu, The reductivity of spherical curves, Topology and its Applications 196 (2015).

# §3. 2-gons & 3-gons

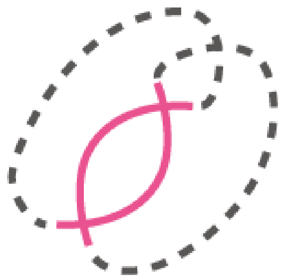


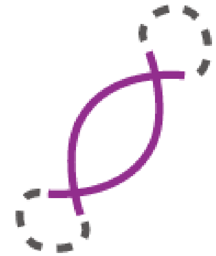
Some types of regions tell us upper bounds of reductivity

# 2-gons (bigons) & 3-gons (trigons)

## There are two types of 2-gons:

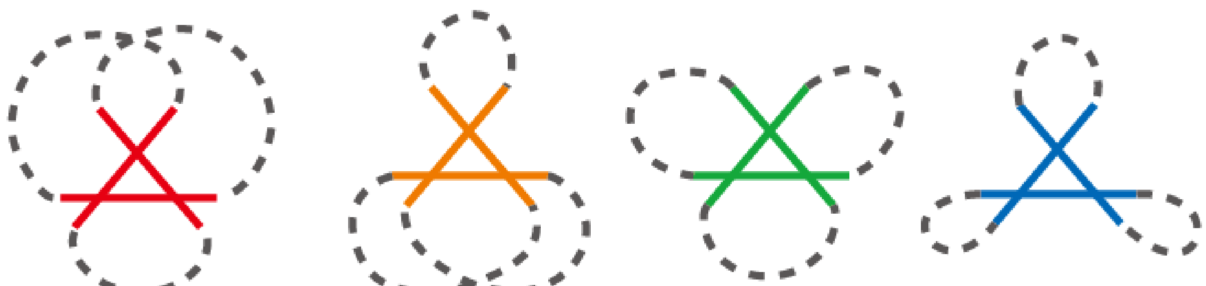
incoherent 2-gon





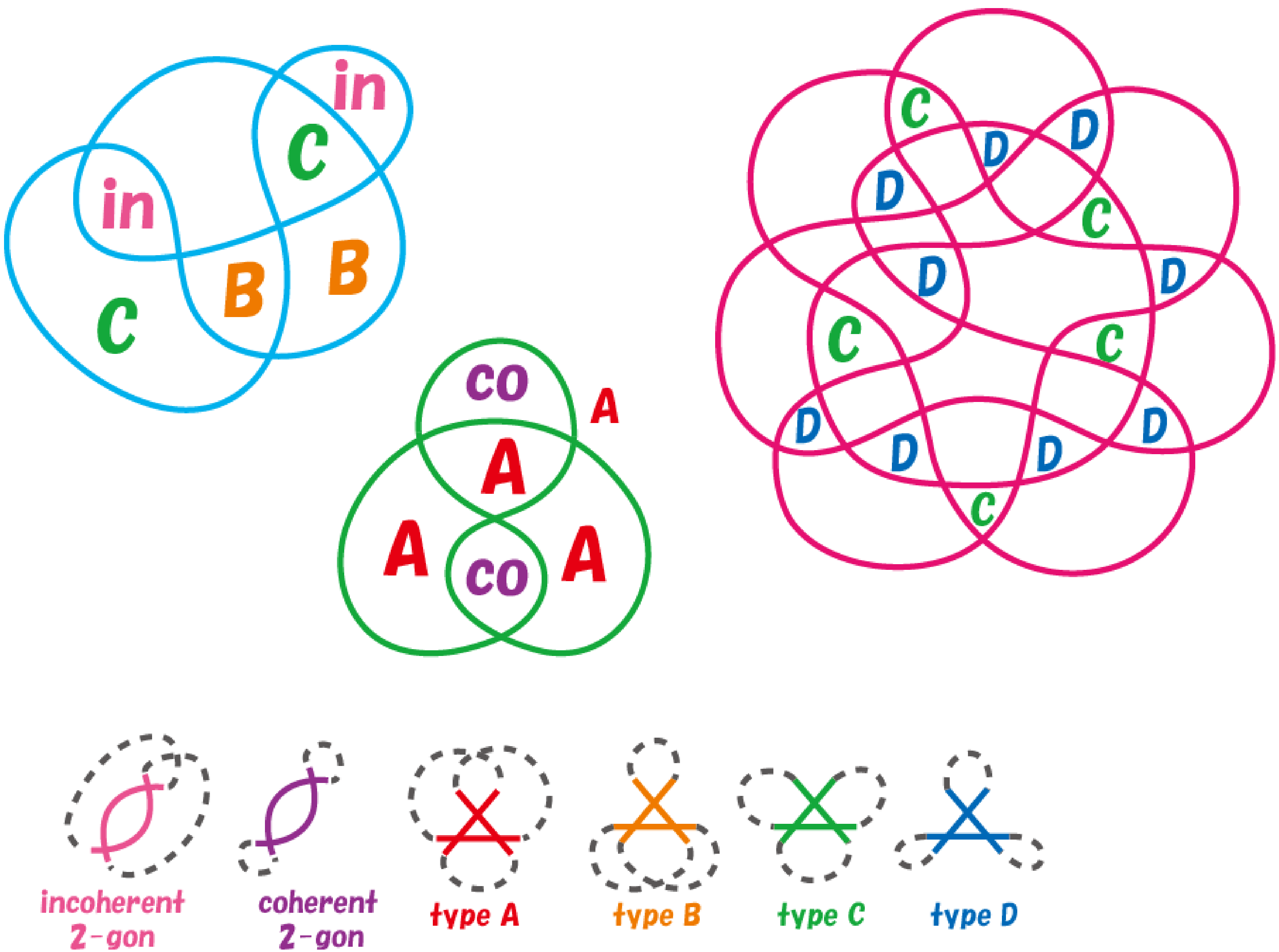
coherent 2-gon

## There are four types of 3-gons:



type A type B type C type D

# 2-gons & 3-gons



in
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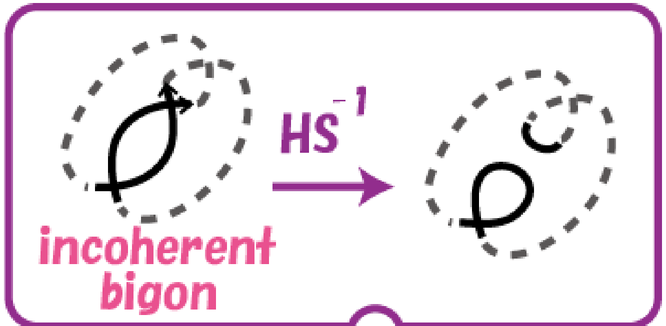
D
C
C
D
D
D
D
C
incoherent 2-gon
coherent 2-gon
type A
type B
type C
type D

# 2-gons




## Proposition 2

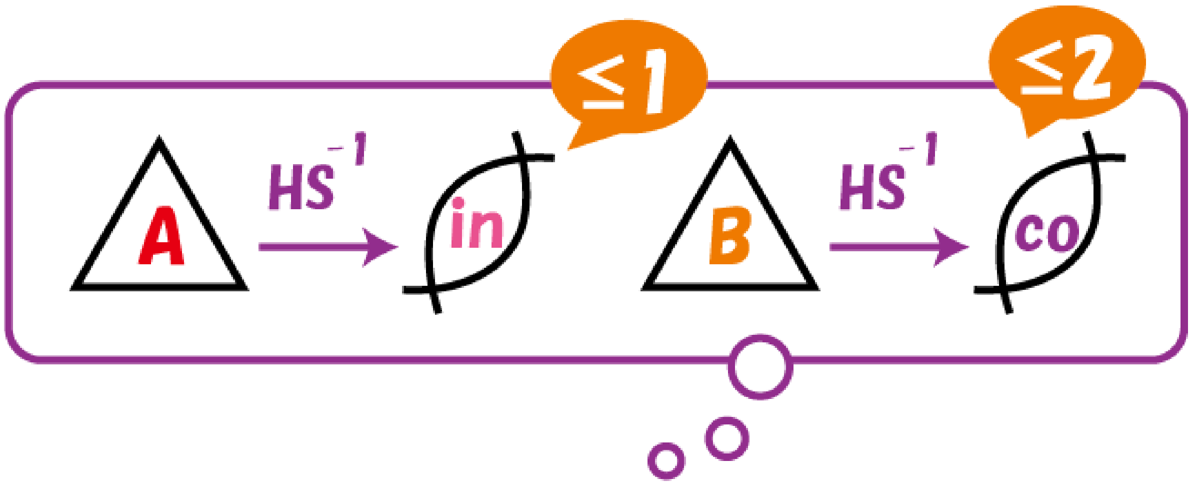
If $P$ has an incoherent 2-gon, then $r(P) \leq 1$.

If $P$ has a coherent 2-gon, then $r(P) \leq 2$.

coherent bigon $\xrightarrow{HS^{-1}}$ incoherent bigon

# 3-gons

## Proposition 3

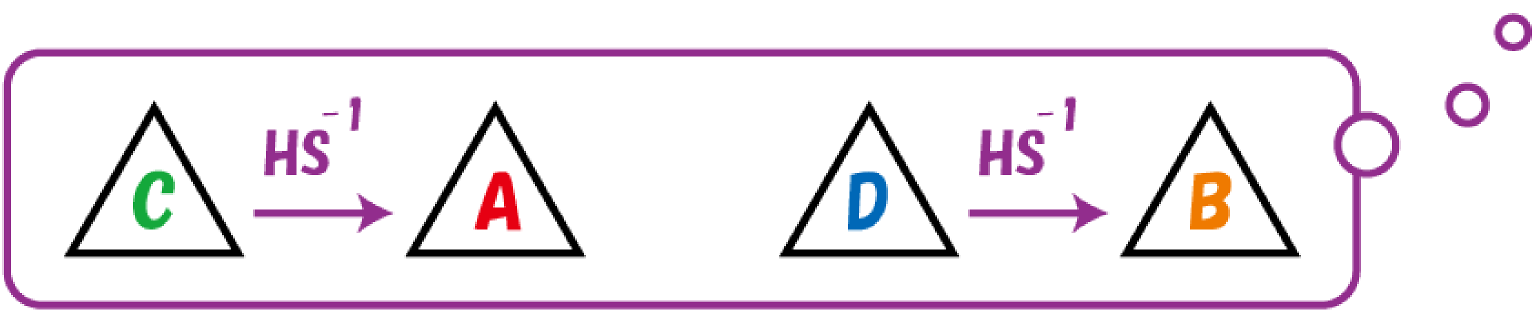



If P has a 3-gon of type A, then $r(P) \leq 2$.

If P has a 3-gon of type B, then $r(P) \leq 3$.

If P has a 3-gon of type C, then $r(P) \leq 3$.

If P has a 3-gon of type D, then $r(P) \leq 4$.

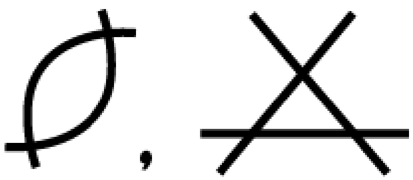


## § 4. Unavoidable sets

Every reduced knot projection must have one of the parts 

# AST' s theorem

**Theorem (Adams-Shinjo-Tanaka)**
**Every reduced knot projection has 2-gon or 3-gon.**

i.e., {  } is an **unavoidable set** for a reduced knot projection.

**Reference**: C. C. Adams, R. Shinjo and K. Tanaka, Complementary regions of knot and link diagrams, Ann. Comb. 15 (2011), 549-563.

# Proof of AST's theorem

$P$: a reduced knot projection

$c_n$: the number of $n$-gons of $P$

**Euler's characteristic**

$$v - e + f = 2$$

$$v = \sum_k \frac{k}{4} c_k \qquad e = \sum_k \frac{k}{2} c_k \qquad f = \sum_k c_k$$

$$\rightarrow \quad 2c_2 + c_3 = 8 + c_5 + 2c_6 + 3c_7 + \cdots$$

# Proof of Theorem 1 (reductivity is four or less)

If P is reducible, then $r(P)=0$.

(by definition)

If P is reduced, P has a 2-gon or 3-gon.

(by AST's theorem)

If P has a 2-gon, then $r(P) \leq 2$.

(by Proposition 2)

If P has a 3-gon, then $r(P) \leq 4$.

(by Proposition 3)

# Unavoidable sets

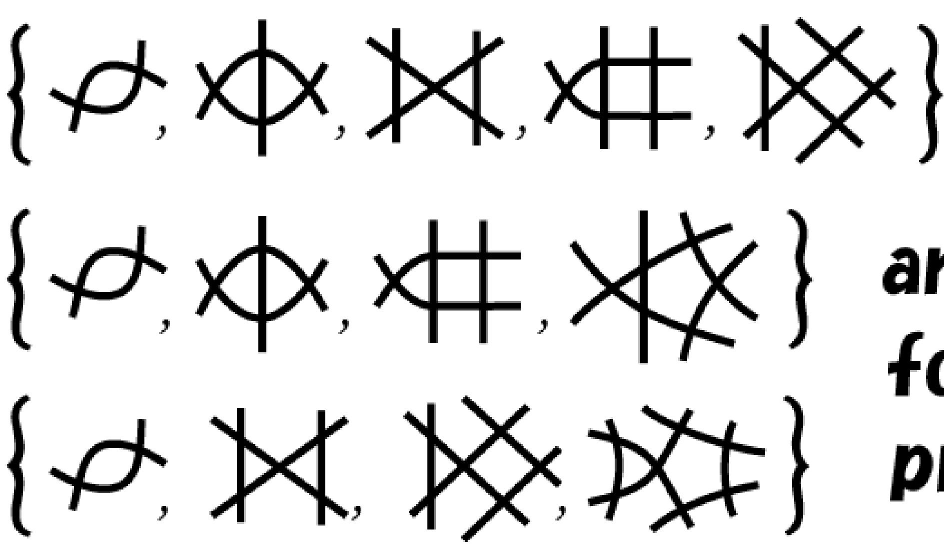
By $2C_2 + C_3 = 8 + C_5 + 2C_6 + 3C_7 + \cdots$

and the "discharding method",

we have:

from graph theory
(four color theorem)

## Proposition 4

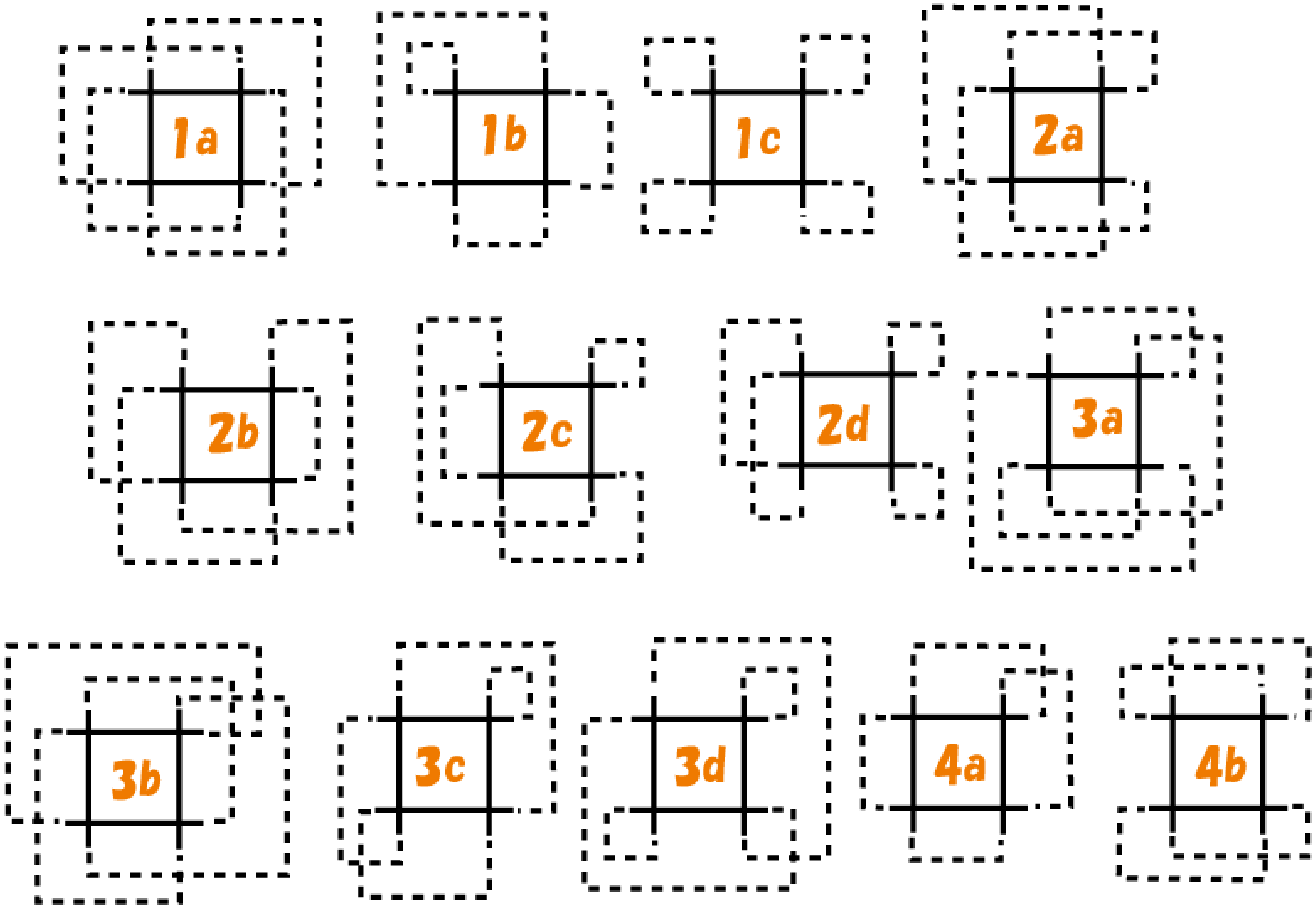


are unavoidable sets for a reduced knot projection.

# §5. 4-gons

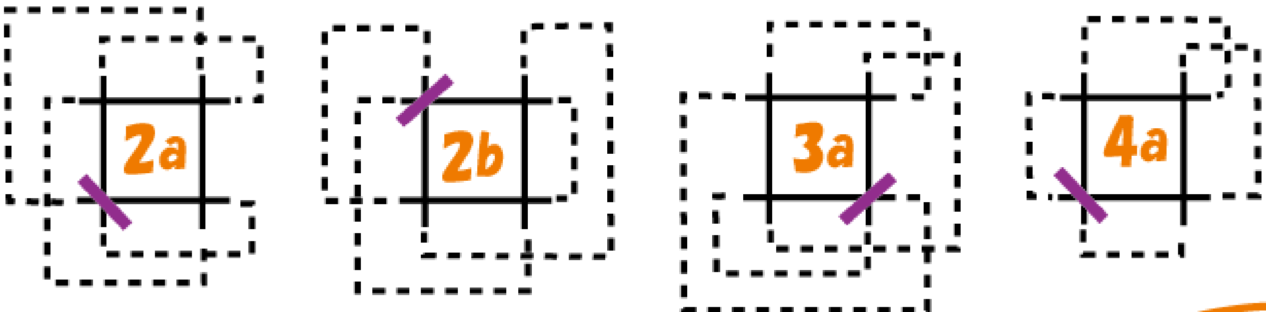
# 4-gons

## There are 13 types of 4-gons:



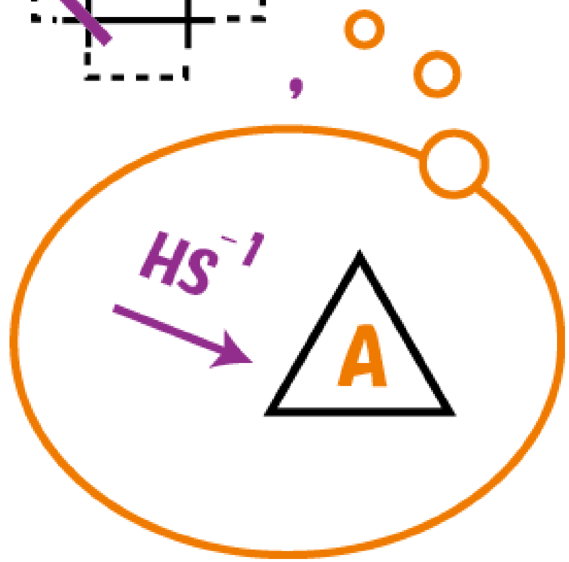
## Proposition 5 (Onoda-S)

If a knot projection P has one of



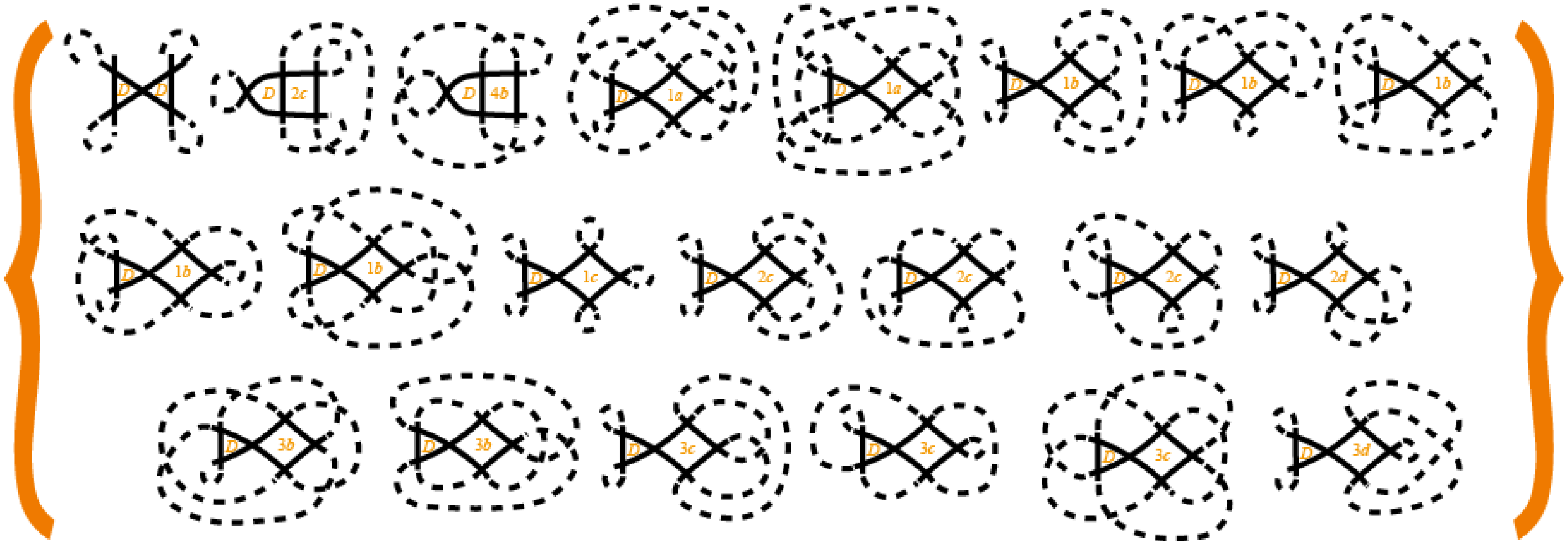

,

then $r(P) \leq 3$.



# Unavoidable set for P with r(P)=4

## Theorem 6 (Onoda-S)



is an unavoidable set for a knot projection with reductivity four.

Reference: Y. Onoda and A. Shimizu, The reductivity of spherical curves Part II: 4-gons, preprint (arXiv:1603.07811).

# §6. Future studies

# Future studies

Create a knot projection P such that $r(P)=4$ (?) using parts of unavoidable sets.
(We need more unavoidable sets..)

Classify 5-gons and get another unavoidable set for P s.t. $r(P)=4$ using chord diagrams and computer.

# Thank you

for listening!