Computation of string operations using rational homotopy theory

若月駿

東大数理 D1 (日本学術振興会特別研究員 DC1)

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Notation

Remark

In this talk,

- coefficients in $\mathbb{Q}$
- spaces are 1-connected
- "commutative" means "graded commutative"

What is string topology?

Chas-Sullivan '99, Cohen-Godin '05

M: an oriented connected closed m-manifold

$LM = \mathrm{Map}(S^1, M)$: the free loop space on M

$$\begin{cases} \mu\colon & H_*(LM)^{\otimes 2} \to H_{*-m}(LM) & \text{(the loop product)} \\ \delta\colon & H_*(LM) \to (H_*(LM)^{\otimes 2})_{*-m} & \text{(the loop coproduct)} \end{cases}$$

String topology: Study these operations

Construction of string operations

μ is constructed by mixing

- intersection product on the homology of a manifold
  $H_*(M)^{\otimes 2} \to H_{*-m}(M)$
- Pontrjagin product defined by the composition of based loops
  $H_*(\Omega M)^{\otimes 2} \to H_*(\Omega M)$
  ($\Omega M = \mathrm{Map}_*(S^1, M)$: the based loop space on M)

intersection product

The intersection product is defined as

$$\Delta^!\colon H_*(M)^{\otimes 2} \xrightarrow{\cong} H^{m-*}(M)^{\otimes 2} \xrightarrow{\Delta^*} H^{2m-*}(M) \xrightarrow{\cong} H_{*-m}(M)$$

using Poincaré duality of M

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Generalization of string operations

Félix-Thomas '09

Generalized μ, δ for Gorenstein spaces

Gorenstein space

Gorenstein space: a generalization of a space satisfying Poincaré duality

Examples:

- oriented connected closed manifolds
- classifying spaces of connected Lie groups

Triviality of string operations

Theorem

1. (Tamanoi '10)
   M: a connected oriented closed manifold $\Rightarrow \delta$ is almost trivial
   ($\delta = 0$ if $\chi(M) = 0$)
2. (Félix-Thomas '09)
   $M = BG$: the classifying space of a connected Lie group $\Rightarrow \mu = 0$
3. (Naito '13)
   $\dim(\pi_{\mathrm{even}}(M) \otimes \mathbb{Q}) < \dim(\pi_{\mathrm{odd}}(M) \otimes \mathbb{Q}) < \infty$ and
   minimal Sullivan model of M is pure
   $\Rightarrow \delta = 0$

1 and 2 are "dual" to each other

Main result

- Explicit descliption of μ, δ using rational homotopy theory when $\dim(\pi_*(M) \otimes \mathbb{Q}) < \infty$
- (partial) generalization of ❶ ❷ ❸ in the above theorem using the explicit description

"Theorem" (W.)

M: Gorenstein space with $\dim(\pi_*(M) \otimes \mathbb{Q}) < \infty$

1. $F \to M \to K(\mathbb{Z}, 2n+1)$: fibration (+ some condition) $\Rightarrow \delta = 0$ for M
2. $K(\mathbb{Z}, 2n) \to M \to B$: fibration (+ some condition) $\Rightarrow \mu = 0$ for M
3. $\dim(\pi_{\mathrm{odd}}(M) \otimes \mathbb{Q}) > \dim(\pi_{\mathrm{even}}(M) \otimes \mathbb{Q}) \Rightarrow \delta = 0$ for M

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Method

based on rational homotopy theory

- Explicit description:
  Compute very easy examples and generalize them
- Triviality:
  Compute many examples by the explicit description using a computer

DGA

Definition

A *Differential Graded Algebra(DGA)* is a pair (A, d) of a graded algebra $A = \{A^n\}_{n \geq 0}$ and a linear map $d\colon A \to A$ satisfying $d^2 = 0$ and the Leibniz rule $d(ab) = da \cdot b + (-1)^{|a|} a \cdot db$

Example

the singular cochain algebra $C^*(X) = C^*(X; \mathbb{Q})$ is a DGA
$C^*(X)$ is non-commutative, but $H^*(X) = H^*(C^*(X))$ is commutative

quasi-isomorphism

Definition

- a DGA homomorphism $f\colon (A,d) \xrightarrow{\simeq^{\mathrm{q}}} (B,d)$ is a *quasi-isomorphism* $\overset{\mathrm{def}}{\Longleftrightarrow}$ $H^*(f)\colon H^*(A,d) \xrightarrow{\cong} H^*(B,d)$ is an isomorphism
- two DGA's $(A,d),(B,d)$ are *quasi-isomorphic* $\overset{\mathrm{def}}{\Longleftrightarrow}$ there is a sequence of quasi-isomorphisms of the form: $(A,d) \xleftarrow{\simeq^{\mathrm{q}}} (C_1,d) \xrightarrow{\simeq^{\mathrm{q}}} (C_2,d) \xleftarrow{\simeq^{\mathrm{q}}} \cdots \xleftarrow{\simeq^{\mathrm{q}}} (C_n,d) \xrightarrow{\simeq^{\mathrm{q}}} (B,d)$

polynomial differential form

Theorem(Sullivan)

X: space
There is a commutative DGA $A^*_{\mathrm{PL}}(X)$ which is quasi-isomorphic to $C^*(X)$

An element $\omega \in A^*_{\mathrm{PL}}(X)$ is called a *polynomial differential form*, which is a "simplicial version" of a differential form (with polynomial coefficients).

$A^*_{\mathrm{PL}}(X)$ is easier than $C^*(X)$ because of commutativity, but still difficult to compute by hand
$\rightarrow$ consider Sullivan models

Sullivan algebra

$\wedge V = \mathrm{Polynomial}(V^{\mathrm{even}}) \otimes \mathrm{Exterior}(V^{\mathrm{odd}})$
free commutative graded algebra generated by graded $\mathbb{Q}$-module V

Definition(Sullivan algebra)

A (1-connected) *Sullivan algebra* is a DGA of the form $(\wedge V, d)$ with $V = \{V^n\}_{n \geq 2}$.

The multiplication on a Sullivan algebra is very easy,
but the differential on it can be difficult

Sullivan model

Theorem(Sullivan)

(A, d): commutative DGA with $H^0(A, d) = \mathbb{Q}, H^1(A, d) = 0$
There is a Sullivan algebra $(\wedge V, d)$ and a quasi-isomorphism

$$\varphi\colon (\wedge V, d) \xrightarrow{\simeq^{\mathrm{q}}} (A, d)$$

φ (or $(\wedge V, d)$): *Sullivan model* of (A, d)
a *Sullivan model* of a space X is a Sullivan model of $A^*_{\mathrm{PL}}(X)$

Examples of Sullivan models (1)

$\wedge(x_1, \dots x_n) = \wedge(\mathrm{span}_{\mathbb{Q}}(x_1, \dots, x_n))$

Example1

1. $(\wedge(x), dx = 0)$ with $|x| = 2n + 1$ is a Sullivan model of the $(2n + 1)$-dimensional sphere S^{2n+1}
2. $(\wedge(x, y), dx = 0, dy = x^2)$ with $|x| = 2n, |y| = 4n - 1$ is a Sullivan model of the $2n$-dimensional sphere S^{2n}

Examples of Sullivan models (2)

Example1

1. $(\wedge(x), dx = 0)$ is a Sullivan model of S^{2n+1}
2. $(\wedge(x, y), dx = 0, dy = x^2)$ is a Sullivan model of S^{2n}

$LX = Map(S^1, X)$ the free loop space on a space X

Example2

1. $(\wedge(x, \bar{x}), dx = d\bar{x} = 0)$ with $|x| = 2n+1, |\bar{x}| = 2n$
   is a Sullivan model of LS^{2n+1}
2. $(\wedge(x, y, \bar{x}, \bar{y}), dx = 0, dy = x^2, d\bar{x} = 0, d\bar{y} = -2x\bar{x})$ with
   $|x| = 2n, |y| = 4n-1, |\bar{x}| = 2n-1, |\bar{y}| = 4n-2$
   is a Sullivan model of LS^{2n}

Demonstration

We give demonstrations of our computer program.

Thank you for your attention!