On the Fuchsian locus of $\mathrm{PSL}_n(\mathbb{R})$-Hitchin components for a pair of pants

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1. Introduction

2. The Bonahon-Dreyer's parametrization

3. Parameterizing the Fuchsian locus

Introduction.

Teichmüller components

- S: a compact connected orientable surface with $\chi(S) < 0$.
- $\mathcal{M}(S)$: the set of complete finite-volumed Riemannian metrics on S.
- $\mathrm{Diff}_0(S)$: the identity component of the diffeomorphism group of S.
- $\mathscr{T}(S) = \mathcal{M}(S)/\mathrm{Diff}_0(S)$: the **Teichmüller space** for S.

The Teichmüller space is identified with a space of representations.

$$\mathscr{T}(S) = \{\rho \in \mathrm{Hom}(\pi_1(S), \mathrm{PSL}_2(\mathbb{R})) \mid \rho \text{ is discrete and faithful}\}/\mathit{Conj}.$$

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Theorem (Goldman '88)

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- The component $\mathrm{Fuch}_2(S)$ is called **Teichmüller component**.
- A representation $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$ is called a **Fuchsian representation** if ρ is discrete and faithful. (i.e. $[\rho] \in \mathrm{Fuch}_2(S)$).
- $\mathcal{F}_2(S)$: the set of Fuchsian representations.

$\mathrm{PSL}_n(\mathbb{R})$-Hitchin components (1)

- The $\mathrm{PSL}_n(\mathbb{R})$-**representation variety** for $\pi_1(S)$ is the set of $\mathrm{PSL}_n(\mathbb{R})$-representations of $\pi_1(S)$ with the compact open topology.

$$\mathcal{R}_n(S) = \mathrm{Hom}(\pi_1(S), \mathrm{PSL}_n(\mathbb{R})).$$

- $\mathrm{PSL}_n(\mathbb{R}) \curvearrowright \mathcal{R}_n(S)$: the conjugate action.
- The $\mathrm{PSL}_n(\mathbb{R})$-**character variety** for $\pi_1(S)$ is the GIT-quotient space

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Theorem (Hitchin '92)

Suppose that S is closed. For $n \geq 3$

$$\#\text{ of components of } \mathcal{X}_n(S) = \begin{cases} 3 & \textit{if } \text{n}: \textit{odd} \\ 6 & \textit{if } \text{n}: \textit{even}. \end{cases}$$

$\mathrm{PSL}_n(\mathbb{R})$-Hitchin components (2)

- $\iota_n : \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_n(\mathbb{R})$: the irreducible representation.
- $(\iota_n)_* : \mathcal{X}_2(S) \to \mathcal{X}_n(S) : (\iota_n)_*([\rho]) = [\iota_n \circ \rho]$.

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Definition

The $\mathrm{PSL}_n(\mathbb{R})$-Hitchin component for S, denoted by $\mathrm{Hit}_n(S)$, is the connected component of $\mathcal{X}_n(S)$ containing $\mathrm{Fuch}_n(S) = (\iota_n)_*(\mathrm{Fuch}_2(S))$.

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- We call $\rho \in \mathcal{R}_n(S)$ a **Hitchin representation** if $[\rho] \in \mathrm{Hit}_n(S)$.
- $\mathcal{H}_n(S)$: the set of Hitchin representations.
- $\mathrm{Fuch}_n(S)$: the **Fuchsian locus**.
- $\iota_n \circ \rho \in \mathcal{R}_n(S)$ $(\rho \in \mathcal{F}_2(S))$: an n-**Fuchsian representation**.
- $\mathcal{F}_n(S)$: the set of n-Fuchsian representations.

The Bonahon-Dreyer's parametrization

- $\mathcal{L}$: a geodesic maximal oriented lamination on S with finite leaves.
- $h_1, \cdots, h_s$: biinfinite leaves in $\mathcal{L}$.
- $g_1, \cdots, g_t$: closed leaves in $\mathcal{L}$.
- $T_1, \cdots, T_u$: ideal triangles in $S \setminus \mathcal{L}$.

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Theorem (Bonahon-Dreyer '14)

There exists an onto-homeomorphism

$$\Phi_{\mathcal{L}} : \mathrm{Hit}_n(S) \to \mathbb{R}^N$$
$$\Phi_{\mathcal{L}}([\rho]) = (\tau^{\rho}_{abc}(\widetilde{T}_i, v_i), \cdots, \sigma^{\rho}_d(h_j), \cdots, \sigma^{\rho}_e(g_k), \cdots).$$

where τ^{ρ}_{abc}, σ^{ρ}_d are the triangle, shearing invariant defined by Bonahon-Dreyer. (We will define later.)

Main result

Goal: To describe $\mathrm{Fuch}_n(S)$ explicitly by using the Bonahon-Dreyer's parametrization for a pair of pants.

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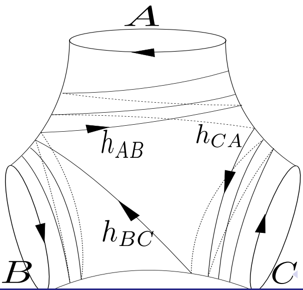
- P: a pair of pants.
- $\mathcal{L}$: the geodesic maximal lamination on P in the figure below.
- $\rho_n \in \mathcal{F}_n(P)$: any n-Fuchsian representation of $\pi_1(P)$.

Theorem (I.)

We can explicitly compute $\Phi_{\mathcal{L}}([\rho_n])$.



The Bonahon-Dreyer's parametrization.

Anosov property of Hitchin representations

- A representation $\rho : \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$ is called an **Anosov representation** if ρ lifts to an $\mathrm{SL}_n(\mathbb{R})$-representation whose flat associate bundle $T^1\widetilde{S} \times_\rho \mathbb{R}^n$ satisfies some dynamical property.
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Theorem (Labourie '06, Fock-Goncharov '06)

Let $\rho : \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$ be a Hitchin representation. Then there exists a unique continuous ρ-equivariant map $\xi_\rho : \partial_\infty \widetilde{S} \to \mathrm{Flag}(\mathbb{R}^n)$ with the hyperconvexity and positivity.

- We call ξ_ρ **flag curve**. (Anosov map, limit map.)

Construction of the Bonahon-Dreyer's parametrization.

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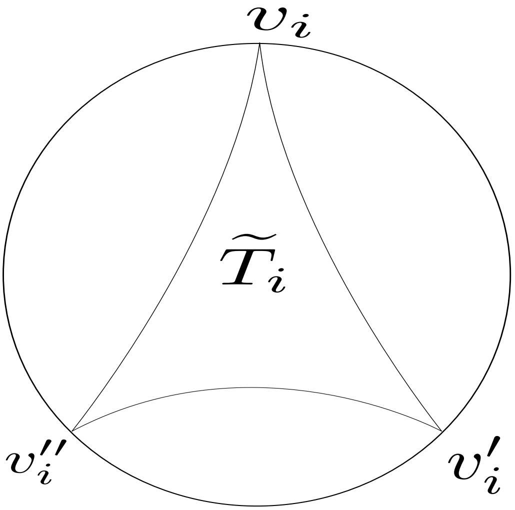
- $\rho \in \mathcal{H}_n(S) \xrightarrow{1:1} \xi_\rho \to \tau^{\rho}_{pqr}, \sigma^{\rho}_{p}$.
- Flag curves are characterized by the invariants $\tau^{\rho}_{pqr}, \sigma^{\rho}_{p}$.

Triangle invariant



Construction of the BD coordinate(Triangle invariant)

- ρ, ξ_ρ: a Hitchin representation and its flag curve.
- T_i: an ideal triangle in $S \setminus \mathcal{L}$.
- $\widetilde{T}_i$: a lifting of T_i in $\widetilde{S}$.
- v, v', v'': ideal vertices of $\widetilde{T}_i$.

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- p, q, r: integers s.t. $p, q, r \geq 1$ and $p + q + r = n$.
- We choose nonzero elements $e^{(i)} \in \bigwedge^{(i)} \xi_\rho(v)^{(i)}, f^{(i)} \in \bigwedge^{(i)} \xi_\rho(v')^{(i)}, g^{(i)} \in \bigwedge^{(i)} \xi_\rho(v'')^{(i)}$.

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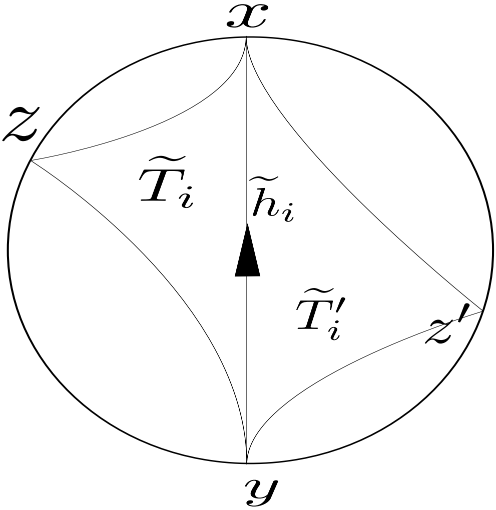
Definition (Triangle invariant)

$$\tau^\rho_{pqr}(\widetilde{T}_i, v) = \log \frac{X(p+1, q, r-1)}{X(p-1, q, r+1)} \cdot \frac{X(p, q-1, r+1)}{X(p, q+1, r-1)} \cdot \frac{X(p-1, q+1, r)}{X(p+1, q-1, r)}$$

where $X(p, q, r) = e^{(p)} \wedge f^{(q)} \wedge g^{(r)}$

Shearing invariant



Construction of the BD coordinate(Shearing invariant)

- $h_i \in \mathcal{L}$: a biinfinite leaf in $\mathcal{L}$.
- T, T': the ideal triangle which are on the left, right of h_i respectively.
- $\widetilde{h}_i$: a lifting of h_i.
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Definition (Shearing invariant)

$$\sigma_p^\rho(h_i) = \log -\frac{Y(p)}{Y'(p)} \cdot \frac{Y'(p-1)}{Y(p-1)}$$

where $Y(i) = e^{(i)} \wedge f^{(n-i-1)} \wedge g^{(1)}$ and $Y'(i) = e^{(i)} \wedge f^{(n-i-1)} \wedge g'^{(1)}$

Parameterizing the Fuchsian locus.

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Then, the following map is an onto-homeomorphism:

$$\begin{aligned}\Phi_{\mathcal{L}} : \mathrm{Hit}_n(P) &\to \mathbb{R}^N, \\ \Phi_{\mathcal{L}}([\rho]) &= (\sigma_p^\rho(h_{AB}), \cdots, \sigma_p^\rho(h_{BC}), \cdots, \sigma_p^\rho(h_{CA}), \cdots \\ &\quad \cdots, \tau_{pqr}^\rho(\widetilde{T}_0, v_0), \cdots, \tau_{pqr}^\rho(\widetilde{T}_1, v_1), \cdots).\end{aligned}$$

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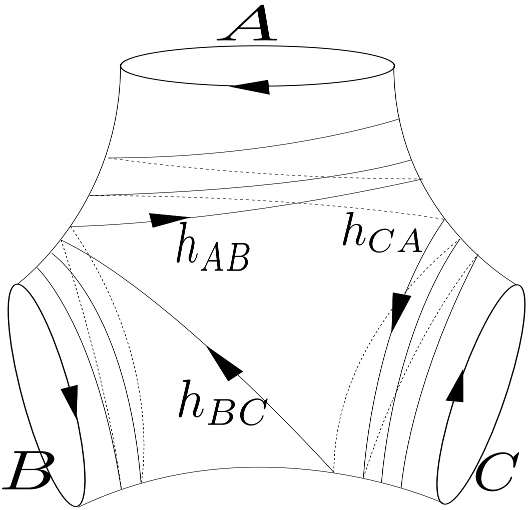
Remark.

The shearing invariants of the closed leaves in boundary are determined by other invariants.

Lamination



Outline of the computation

Goal: To describe $\mathrm{Fuch}_n(S)$ explicitly by using the Bonahon-Dreyer's parametrization for a pair of pants. In particular, we compute $\Phi_{\mathcal{L}}([\rho_n])$ for any n-Fuchsian representation.

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Outline.

1. We parameterize $\rho \in \mathcal{F}_2(P)$ by the hyperbolic length of the boundary components.
2. Describe the flag curve ξ_{ρ_n} of $\rho_n = \iota_n \circ \rho$.
3. Compute the invariants $\sigma_p^{\rho_n}, \tau_{pqr}^{\rho_n}$.

1. Parameterizing Fuchsian representations

- $\mathbf{m} \in \mathscr{T}(P)$: a hyperbolic structure on P.
- $\mathcal{S}$: the set of simple closed curves.
- $l_{\mathbf{m}} : \mathcal{S} \to \mathbb{R}_{>0}$: the length function associated to $\mathbf{m}$.

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Proposition

The following map is a diffeomorphism.

$$\begin{aligned} \mathscr{T}(P) &\to \mathbb{R}^3_{>0}, \\ \mathbf{m} &\mapsto (l_{\mathbf{m}}(A), l_{\mathbf{m}}(B), l_{\mathbf{m}}(C)). \end{aligned}$$

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- $\mathbf{m} = (l_A, l_B, l_C)$: the hyperbolic length of the boundary components.
- a, b, c: the homotopy classes of A, B, C.
- $\pi_1(P) =< a, b, c \mid abc = 1 >$: a presentation.
- ρ: a Fuchsian representation.

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Proposition (I.)

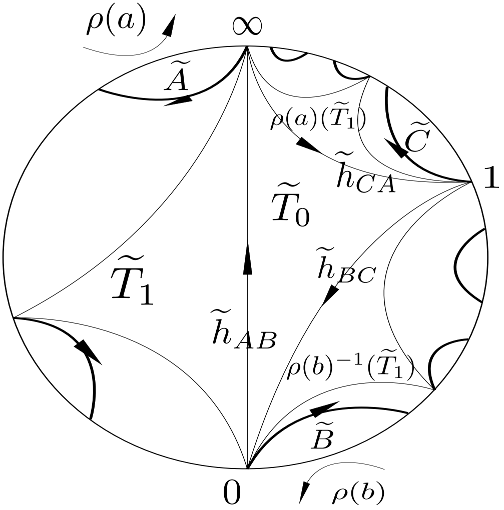
If ρ is a Fuchsian representation associated to $\mathbf{m}$, then ρ is conjugate to the following representation.

$$\rho(a) = \begin{bmatrix} \alpha & \alpha\beta\gamma + \alpha^{-1} \\ 0 & \alpha^{-1} \end{bmatrix}, \quad \rho(b) = \begin{bmatrix} \gamma & 0 \\ -\beta^{-1} - \gamma^{-1} & \gamma^{-1} \end{bmatrix}$$

where $\alpha, \beta, \gamma : \mathbb{R}^3_{>0} \to \mathbb{R}_{>0}$ is defined by

$$\alpha(l_A, l_B, l_C) = e^{l_A/2}, \quad \beta(l_A, l_B, l_C) = e^{(l_C - l_A)/2}, \quad \gamma(l_A, l_B, l_C) = e^{-l_B/2}.$$

Developing image



2. Describing the flag curve $\xi_\rho : \partial_\infty \widetilde{S} \to \mathrm{Flag}(\mathbb{R}^n)$

- $\rho_n = \iota_n \circ \rho \in \mathcal{F}_n(P)$: an n-Fuchsian representation.
- $\mathbf{Dev}_\rho : \widetilde{P} \to \mathbb{H}^2$: the developing map associated to ρ.
- The developing map $\mathbf{Dev}_\rho$ gives the embedding $\partial_\infty P \to \partial_\infty \mathbb{H}^2$.

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- $V = Span_{\mathbb{R}} < X^{n-1}, X^{n-2}Y, \cdots, Y^{n-1} >$: an n-dim. $\mathbb{R}$-vec. sp.

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We define a subspace $W^{(i)}(z)$ of V for $z \in \partial_\infty \mathbb{H}^2$ as follows.

$$W^{(i)}(z) = \text{the set of polynomials which can be divided by } \begin{cases} (zX+Y)^{n-i} & (z \neq \infty) \\ X^{n-i} & (z = \infty). \end{cases}$$

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$$\begin{array}{rcl} \xi_{\rho_n} : \partial_\infty P \xrightarrow{\mathbf{Dev}_\rho} \partial_\infty \mathbb{H}^2 & \to & \mathrm{Flag}(\mathbb{R}^n) \\ z & \mapsto & (W^{(i)}(z))_{i=0,\cdots,n} \end{array}$$

Computation (1)

$$\boxed{\tau_{pqr}^{\rho_n}(\widetilde{T}_0, \infty)}$$

Set $E = \xi_{\rho_n}(\infty), F = \xi_{\rho_n}(1), G = \xi_{\rho_n}(0)$.

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Flags.

$$\begin{aligned}
E^{(p)} &= \mathrm{Span}_{\mathbb{R}} < X^{n-1}, X^{n-2}Y, \cdots, X^{n-p}Y^{p-1} >, \\
F^{(q)} &= \mathrm{Span}_{\mathbb{R}} < (X+Y)^{n-q}X^{q-1}, (X+Y)^{n-q}X^{q-2}Y, \cdots, (X+Y)^{n-q}Y^{q-1} > \\
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\end{aligned}
$$

Nonzero elements.

$$
\begin{aligned}
e^{(p)} &= X^{n-1} \wedge X^{n-2}Y \wedge \cdots \wedge X^{n-p}Y^{p-1}, \\
f^{(q)} &= (X+Y)^{n-q}X^{q-1} \wedge (X+Y)^{n-q}X^{q-2}Y \wedge \cdots \wedge (X+Y)^{n-q}Y^{q-1}, \\
g^{(r)} &= Y^{n-1} \wedge XY^{n-2} \wedge \cdots \wedge X^{r-1}Y^{n-r}.
\end{aligned}
$$

Computation (2)

$$\tau^{\rho}_{pqr}(\widetilde{T}_0, \infty) = \log \frac{X(p+1,q,r-1)}{X(p-1,q,r+1)} \cdot \frac{X(p,q-1,r+1)}{X(p,q+1,r-1)} \cdot \frac{X(p-1,q+1,r)}{X(p+1,q-1,r)}$$

where $X(p,q,r) = e^{(p)} \wedge f^{(q)} \wedge g^{(r)}$.

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where $X(p,q,r) = e^{(p)} \wedge f^{(q)} \wedge g^{(r)}$.

Fix a basis $b_1 = X^{n-1}, b_2 = X^{n-2}Y, \cdots, b_n = Y^{n-1}$ of V. Then

$$(X+Y)^{n-q}X^{q-k}Y^{k-1} = \binom{n-q}{0} b_k + \binom{n-q}{1} b_{k+1} + \cdots + \binom{n-q}{n-q} b_{n-q+k}.$$

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Notation.

$$\binom{n}{p} = \begin{cases} \dfrac{n!}{p!(n-p)!} & (0 \leq p \leq n) \\ 0 & (otherwise). \end{cases}$$

Computation (3)

$$A_{pqr}(T_0) = \begin{pmatrix} \binom{p+r}{0} & \binom{p+r}{-1} & \cdots & \binom{p+r}{-q+1} \\ \binom{p+r}{1} & \binom{p+r}{0} & \cdots & \binom{p+r}{-q+2} \\ \vdots & \vdots & \vdots & \vdots \\ \binom{p+r}{p+r} & \binom{p+r}{p+r-1} & \cdots & \binom{p+r}{0} \\ \binom{p+r}{p+r+1} & \binom{p+r}{p+r} & \cdots & \binom{p+r}{1} \\ \vdots & \vdots & \vdots & \vdots \\ \binom{p+r}{n-1} & \binom{p+r}{n-2} & \cdots & \binom{p+r}{p+r} \end{pmatrix},$$

$$X(p,q,r) = \begin{vmatrix} \mathrm{Id}_p & & 0 \\ & A_{pqr}(T_0) & \\ 0 & & \mathrm{Id}_r \end{vmatrix} = \begin{vmatrix} \binom{p+r}{p} & \cdots & \binom{p+r}{p-q+1} \\ \vdots & \vdots & \vdots \\ \binom{p+r}{p+q-1} & \cdots & \binom{p+r}{p} \end{vmatrix}$$

and $X(p,0,r) = 1$ for all p, r.

Result 1

Triangle invariant $\tau^{\rho_n}_{pqr}(\widetilde{T}_0, \infty)(p, q, r \geq 1 \ s.t. \ p + q + r = n)$.

$$\tau^{\rho_n}_{pqr}(\widetilde{T}_0, \infty) = \log \frac{X_{T_0}(p+1, q, r-1)}{X_{T_0}(p-1, q, r+1)} \cdot \frac{X_{T_0}(p, q-1, r+1)}{X_{T_0}(p, q+1, r-1)} \cdot \frac{X_{T_0}(p-1, q+1, r)}{X_{T_0}(p+1, q-1, r)}$$

where

$$X_{T_0}(p, q, r) = \begin{vmatrix} \binom{p+r}{p} & \cdots & \binom{p+r}{p-q+1} \\ \vdots & \vdots & \vdots \\ \binom{p+r}{p+q-1} & \cdots & \binom{p+r}{p} \end{vmatrix}$$

if $q \neq 0$ and $X_{T_0}(p, 0, r) = 1$ for all p, r.

Result 2

Triangle invariant $\tau^{\rho_n}_{pqr}(\widetilde{T}_1, \infty)(p, q, r \geq 1 \ s.t. \ p + q + r = n)$.

$$\tau^{\rho_n}_{pqr}(\widetilde{T}_1, \infty) = \log \frac{X_{T_1}(p+1, q, r-1)}{X_{T_1}(p-1, q, r+1)} \cdot \frac{X_{T_1}(p, q-1, r+1)}{X_{T_1}(p, q+1, r-1)} \cdot \frac{X_{T_1}(p-1, q+1, r)}{X_{T_1}(p+1, q-1, r)}$$

where

$$X_{T_1}(p, q, r) = (-1)^{q(r+1)} \begin{vmatrix} \binom{p+q}{p}(-\beta\gamma)^q & \cdots & \binom{p+q}{p-r+1}(-\beta\gamma)^{q+r-1} \\ \vdots & \vdots & \vdots \\ \binom{p+q}{p+r-1}(-\beta\gamma)^{q-r+1} & \cdots & \binom{p+q}{p}(-\beta\gamma)^q \end{vmatrix}$$

if $r \neq 0$ and $X_{T_1}(p, q, 0) = (-1)^q$ for all p, q.

Result 3

Shearing invariant $\sigma_p^{\rho_n}(h_{AB})(1 \leq p \leq n-1)$.

$$\sigma_p^{\rho_n}(h_{AB}) = \log - \frac{Y_{h_{AB}}(p)}{Y'_{h_{AB}}(p)} \cdot \frac{Y'_{h_{AB}}(p-1)}{Y_{h_{AB}}(p-1)}$$

where

$$Y_{h_{AB}}(p) = \binom{n-1}{p} (\beta\gamma)^{n-p-1},$$

$$Y'_{h_{AB}}(p) = (-1)^{n-p-1} \binom{n-1}{p}.$$

Result 4

Shearing invariant $\sigma_p^{\rho_n}(h_{BC})(1 \leq p \leq n-1)$.

$$\sigma_p^{\rho_n}(h_{BC}) = \log -\frac{Y_{h_{BC}}(p)}{Y'_{h_{BC}}(p)} \cdot \frac{Y'_{h_{BC}}(p-1)}{Y_{h_{BC}}(p-1)}$$

where

$$Y_{h_{BC}}(p) = (-1)^{(n-p)p} \begin{vmatrix} \binom{p+1}{0} & \cdots & \binom{p+1}{-n+p+2} & \binom{n-1}{0}\left(\frac{\beta}{\beta+\gamma}\right)^{n-1} \\ \vdots & \vdots & \vdots & \\ \binom{p+1}{n-p-1} & \cdots & \binom{p+1}{1} & \binom{n-1}{n-p-1}\left(\frac{\beta}{\beta+\gamma}\right)^{p} \end{vmatrix}$$

if $p \neq n-1$ and $Y_{h_{BC}}(n-1) = (-1)^{n-1}\binom{n-1}{0}\left(\frac{\beta}{\beta+\gamma}\right)^{n-1}$,

$$Y'_{h_{BC}}(p) = (-1)^{np+n+1} \begin{vmatrix} \binom{p+1}{1} & \cdots & \binom{p+1}{-n+p+3} \\ \vdots & & \vdots \\ \binom{p+1}{n-p-1} & \cdots & \binom{p+1}{1} \end{vmatrix}$$

if $p \neq n-1$ and $Y'_{h_{BC}}(n-1) = (-1)^{n-1}$.

Result 5

Shearing invariant $\sigma_p^{\rho_n}(h_{CA})(1 \leq p \leq n-1)$.

$$\sigma_p^{\rho_n}(h_{CA}) = \log -\frac{Y_{h_{CA}}(p)}{Y'_{h_{CA}}(p)} \cdot \frac{Y'_{h_{CA}}(p-1)}{Y_{h_{CA}}(p-1)}$$

where

$$Y_{h_{CA}}(p) = (-1)^{np} \begin{vmatrix} \binom{n-p}{n-p-1} & \cdots & \binom{n-p}{n-2p} & \binom{n-1}{n-p-1}(\alpha^2\beta\gamma+1)^p \\ \vdots & & \vdots & \vdots \\ \binom{n-p}{n-1} & \cdots & \binom{n-p}{n-p} & \binom{n-1}{n-1}(\alpha^2\beta\gamma+1)^0 \end{vmatrix}$$

if $p \neq 0$ and $Y_{h_{CA}}(0) = 1$,

$$Y'_{h_{CA}}(p) = (-1)^{np} \begin{vmatrix} \binom{n-p}{n-p-1} & \cdots & \binom{n-p}{n-2p} \\ \vdots & \vdots & \vdots \\ \binom{n-p}{n-2} & \cdots & \binom{n-p}{n-p-1} \end{vmatrix}$$

if $p \neq 0$ and $Y'_{h_{CA}}(0) = 1$.

Thank you for your attention!!