On the Fuchsian locus of $\text{PSL}_n(\mathbb{R})$-Hitchin components for a pair of pants

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1 Introduction

2 The Bonahon-Dreyer’s parametrization

3 Parameterizing the Fuchsian locus
Introduction.
Teichmüller components

- $S$: a compact connected orientable surface with $\chi(S) < 0$.
- $\mathcal{M}(S)$: the set of complete finite-volumed Riemannian metrics on $S$.
- $\text{Diff}_0(S)$: the identity component of the diffeomorphism group of $S$.
- $\mathcal{T}(S) = \mathcal{M}(S)/\text{Diff}_0(S)$: the Teichmüller space for $S$.

The Teichmüller space is identified with a space of representations.

$$\mathcal{T}(S) = \{ \rho \in \text{Hom}(\pi_1(S), \text{PSL}_2(\mathbb{R})) \mid \rho \text{ is discrete and faithful} \}/\text{Conj}.$$
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**Theorem (Goldman ’88)**

The subset of $\text{Hom}(\pi_1(S), \text{PSL}_2(\mathbb{R}))/\text{conj}$, denoted by $\text{Fuch}_2(S)$, which consists of discrete, faithful representations is a connected component.

- The component $\text{Fuch}_2(S)$ is called **Teichmüller component**.
- A representation $\rho : \pi_1(S) \to \text{PSL}_2(\mathbb{R})$ is called a **Fuchsian representation** if $\rho$ is discrete and faithful. (i.e. $[\rho] \in \text{Fuch}_2(S)$).
- $\mathcal{F}_2(S)$: the set of Fuchsian representations.
The \( \text{PSL}_n(\mathbb{R}) \)-representation variety for \( \pi_1(S) \) is the set of \( \text{PSL}_n(\mathbb{R}) \)-representations of \( \pi_1(S) \) with the compact open topology.

\[ \mathcal{R}_n(S) = \text{Hom}(\pi_1(S), \text{PSL}_n(\mathbb{R})). \]

- \( \text{PSL}_n(\mathbb{R}) \triangleright \mathcal{R}_n(S) \): the conjugate action.
- The \( \text{PSL}_n(\mathbb{R}) \)-character variety for \( \pi_1(S) \) is the GIT-quotient space

\[ \mathcal{X}_n(S) = \mathcal{R}_n(S)//\text{PSL}_n(\mathbb{R}). \]
The $\text{PSL}_n(\mathbb{R})$-representation variety for $\pi_1(S)$ is the set of $\text{PSL}_n(\mathbb{R})$-representations of $\pi_1(S)$ with the compact open topology.

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$$\mathcal{X}_n(S) = \mathcal{R}_n(S)//\text{PSL}_n(\mathbb{R}).$$

**Theorem (Hitchin ’92)**

Suppose that $S$ is closed. For $n \geq 3$

$$\# \text{ of components of } \mathcal{X}_n(S) = \begin{cases} 3 & \text{if } n: \text{ odd} \\ 6 & \text{if } n: \text{ even}. \end{cases}$$
$\text{PSL}_n(\mathbb{R})$-Hitchin components (2)

- $\iota_n : \text{PSL}_2(\mathbb{R}) \rightarrow \text{PSL}_n(\mathbb{R})$: the irreducible representation.
- $(\iota_n)_* : \mathcal{X}_2(S) \rightarrow \mathcal{X}_n(S) : (\iota_n)_*([\rho]) = [\iota_n \circ \rho]$. 

$\text{Hit}_n(S)$: the set of Hitchin representations.

$\text{Fuch}_n(S)$: the Fuchsian locus.

$\iota_n \circ \rho$:

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PSLₙ(ℝ)-Hitchin components (2)

- \( \iota_n : \text{PSL}_2(ℝ) \to \text{PSL}_n(ℝ) \): the irreducible representation.
- \( (\iota_n)_* : \mathcal{X}_2(S) \to \mathcal{X}_n(S) : (\iota_n)_*[\rho] = [\iota_n \circ \rho] \).

**Definition**

The PSLₙ(ℝ)-Hitchin component for \( S \), denoted by Hitₙ(S), is the connected component of \( \mathcal{X}_n(S) \) containing Fuchₙ(S) = \( (\iota_n)_*(\text{Fuch}_2(S)) \).
\( \iota_n : \text{PSL}_2(\mathbb{R}) \to \text{PSL}_n(\mathbb{R}) \): the irreducible representation.

\((\iota_n)_* : \mathcal{X}_2(S) \to \mathcal{X}_n(S) : (\iota_n)_*([\rho]) = [\iota_n \circ \rho] \).

**Definition**

The \( \text{PSL}_n(\mathbb{R}) \)-Hitchin component for \( S \), denoted by \( \text{Hit}_n(S) \), is the connected component of \( \mathcal{X}_n(S) \) containing \( \text{Fuch}_n(S) = (\iota_n)_*(\text{Fuch}_2(S)) \).

- We call \( \rho \in \mathcal{R}_n(S) \) a **Hitchin representation** if \([\rho] \in \text{Hit}_n(S) \).
- \( \mathcal{H}_n(S) \): the set of Hitchin representations.
- \( \text{Fuch}_n(S) \): the **Fuchsian locus**.
- \( \iota_n \circ \rho \in \mathcal{R}_n(S) (\rho \in \mathcal{F}_2(S)) \): an **\( n \)-Fuchsian representation**.
- \( \mathcal{F}_n(S) \): the set of \( n \)-Fuchsian representations.
The Bonahon-Dreyer’s parametrization

- $\mathcal{L}$: a geodesic maximal oriented lamination on $S$ with finite leaves.
- $h_1, \cdots, h_s$: biinfinite leaves in $\mathcal{L}$.
- $g_1, \cdots, g_t$: closed leaves in $\mathcal{L}$.
- $T_1, \cdots, T_u$: ideal triangles in $S \setminus \mathcal{L}$. 

Theorem (Bonahon-Dreyer '14)

There exists an onto-homeomorphism $\phi: \text{Hit}_n(S) \to \mathcal{R}^N$ where $\phi$ is given by

$$abc(T_1, v_1), \cdots, d(h_j), e(g_k)$$

Bonahon-Dreyer. (We will define later.)
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**Theorem (Bonahon-Dreyer ’14)**

There exists an onto-homeomorphism

$$
\Phi_{\mathcal{L}} : \text{Hit}_n(S) \rightarrow \mathbb{R}^N
$$

$$
\Phi_{\mathcal{L}}([\rho]) = (\tau_{abc}^\rho(\widetilde{T}_i, v_i), \cdots, \sigma_d^\rho(h_j), \cdots, \sigma_e^\rho(g_k), \cdots).
$$

where $\tau_{abc}^\rho$, $\sigma_d^\rho$ are the triangle, shearing invariant defined by Bonahon-Dreyer. (We will define later.)
Main result

Goal: To describe $\text{Fuch}_n(S)$ explicitly by using the Bonahon-Dreyer’s parametrization for a pair of pants.
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Goal: To describe $\text{Fuch}_n(S)$ explicitly by using the Bonahon-Dreyer’s parametrization for a pair of pants.

- $P$: a pair of pants.
- $\mathcal{L}$: the geodesic maximal lamination on $P$ in the figure below.
- $\rho_n \in \mathcal{F}_n(P)$: any $n$-Fuchsian representation of $\pi_1(P)$.

Theorem (I.)

We can explicitly compute $\Phi_\mathcal{L}([\rho_n])$. 

![Diagram of a pair of pants with labeled geodesics and laminations]
The Bonahon-Dreyer’s parametrization.
A representation $\rho : \pi_1(S) \to \text{PSL}_n(\mathbb{R})$ is called an **Anosov representation** if $\rho$ lifts to an $\text{SL}_n(\mathbb{R})$-representation whose flat associate bundle $T^1\widetilde{S} \times_\rho \mathbb{R}^n$ satisfies some dynamical property.

Hitchin representations are Anosov.
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Hitchin representations are Anosov.

**Theorem (Labourie ’06, Fock-Goncharov ’06)**

Let $\rho : \pi_1(S) \to \text{PSL}_n(\mathbb{R})$ be a Hitchin representation. Then there exists a unique continuous $\rho$-equivariant map $\xi_\rho : \partial_\infty \tilde{S} \to \text{Flag}(\mathbb{R}^n)$ with the hyperconvexity and positivity.

- We call $\xi_\rho$ **flag curve**. (Anosov map, limit map.)
Construction of the Bonahon-Dreyer’s parametrization.

Theorem (Bonahon-Dreyer ’14)

There exists an onto-homeomorphism

\[ \Phi_L : \text{Hit}_n(S) \rightarrow \mathbb{R}^N \]

\[ \Phi_L([\rho]) = (\tau^\rho_{abc}(\vec{T}_i, v_i), \ldots, \sigma^\rho_d(h_j), \ldots, \sigma^\rho_e(g_k), \ldots) . \]

where \( \tau^\rho_{abc} \), \( \sigma^\rho_d \) are the triangle, shearing invariant defined by Bonahon-Dreyer. (We will define later.)

- \( \rho \in \mathcal{H}_n(S) \xrightarrow{1:1} \xi_\rho \rightarrow \tau^\rho_{pqr}, \sigma^\rho_p . \)
- Flag curves are characterized by the invariants \( \tau^\rho_{pqr}, \sigma^\rho_p . \)
Triangle invariant

\[ \tilde{T}_i \]

\[ v_i, v'_i, v''_i \]
Construction of the BD coordinate (Triangle invariant)

- \(\rho, \xi_{\rho}\): a Hitchin representation and its flag curve.
- \(T_i\): an ideal triangle in \(S \setminus \mathcal{L}\).
- \(\tilde{T}_i\): a lifting of \(T_i\) in \(\tilde{S}\).
- \(v, v', v''\): ideal vertices of \(\tilde{T}_i\).
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- $\rho, \xi_\rho$: a Hitchin representation and its flag curve.
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- $\tilde{T}_i$: a lifting of $T_i$ in $\tilde{S}$.
- $\nu, \nu', \nu'':$ ideal vertices of $\tilde{T}_i$.
- $p, q, r$: integers s.t. $p, q, r \geq 1$ and $p + q + r = n$.
- We choose nonzero elements 
  
  $e^{(i)} \in \bigwedge^{(i)} \xi_\rho(\nu^{(i)}), f^{(i)} \in \bigwedge^{(i)} \xi_\rho(\nu'^{(i)}), g^{(i)} \in \bigwedge^{(i)} \xi_\rho(\nu''^{(i)}).$
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- $p, q, r$: integers s.t. $p, q, r \geq 1$ and $p + q + r = n$.
- We choose nonzero elements
  
  $e^{(i)} \in \wedge (i) \xi_\rho(v)^{(i)}, f^{(i)} \in \wedge (i) \xi_\rho(v')^{(i)}, g^{(i)} \in \wedge (i) \xi_\rho(v'')^{(i)}$.

**Definition (Triangle invariant)**

\[
\tau_{pqr}^\rho(\tilde{T}_i, v) = \log \frac{X(p+1, q, r-1)}{X(p-1, q, r+1)} \cdot \frac{X(p, q-1, r+1)}{X(p, q+1, r-1)} \cdot \frac{X(p-1, q+1, r)}{X(p+1, q-1, r)}
\]

where $X(p, q, r) = e^{(p)} \wedge f^{(q)} \wedge g^{(r)}$.
Shearing invariant

\[ \tilde{T}_i \]

\[ \tilde{h}_i \]

\[ \tilde{T}_i' \]

\[ z \]

\[ \tilde{z} \]

\[ y \]
Construction of the BD coordinate (Shearing invariant)

- $h_i \in \mathcal{L}$: a biinfinite leaf in $\mathcal{L}$.
- $T, T'$: the ideal triangle which are on the left, right of $h_i$ respectively.
- $\tilde{h}_i$: a lifting of $h_i$.
- $\tilde{T}, \tilde{T}'$: the lifting of $T, T'$ containing $\tilde{h}_i$. 
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- $\tilde{h}_i$: a lifting of $h_i$.
- $\tilde{T}, \tilde{T}'$: the lifting of $T, T'$ containing $\tilde{h}_i$.
- $x, y, z, z'$: ideal vertices of $\tilde{T}, \tilde{T}'$.
- We choose nonzero elements $e^{(i)} \in \wedge^{(i)} \xi_\rho(x)^{(i)}$, $f^{(i)} \in \wedge^{(i)} \xi_\rho(y)^{(i)}$, $g^{(i)} \in \wedge^{(i)} \xi_\rho(z)^{(i)}$, $g'^{(i)} \in \wedge^{(i)} \xi_\rho(z')^{(i)}$.
- $p$: an integer with $1 \leq p \leq n - 1$. 

Definition (Shearing invariant)

$$p(h_i) = \log \frac{Y(p)}{Y'(p)} \frac{Y'(1)}{Y(1)}$$

where

$$Y(i) = e^{(i)} f^{(n - 1)} g^{(1)}, Y'(i) = e^{(i)} f^{(n - 1)} g'^{(1)}.$$
Construction of the BD coordinate (Shearing invariant)

- \( h_i \in \mathcal{L} \): a biinfinite leaf in \( \mathcal{L} \).
- \( T, T' \): the ideal triangle which are on the left, right of \( h_i \) respectively.
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- \( x, y, z, z' \): ideal vertices of \( \tilde{T}, \tilde{T}' \).
- We choose nonzero elements \( e^{(i)} \in \wedge^{(i)} \xi_{\rho}(x)^{(i)}, f^{(i)} \in \wedge^{(i)} \xi_{\rho}(y)^{(i)}, g^{(i)} \in \wedge^{(i)} \xi_{\rho}(z)^{(i)}, g'(i) \in \wedge^{(i)} \xi_{\rho}(z')^{(i)} \).
- \( p \): an integer with \( 1 \leq p \leq n - 1 \).

Definition (Shearing invariant)

\[
\sigma^\rho_p(h_i) = \log \frac{Y(p)}{Y'(p)} \cdot \frac{Y'(p - 1)}{Y(p - 1)}
\]

where \( Y(i) = e^{(i)} \wedge f^{(n-i-1)} \wedge g^{(1)} \) and \( Y'(i) = e^{(i)} \wedge f^{(n-i-1)} \wedge g'(1) \).
Parameterizing the Fuchsian locus.
The Bonahon-Dreyer’s parametrization for our case.

We apply the Bonahon-Dreyer’s parametrization to our case.

- $P$: a pair of pants.
- $L$: the maximal lamination.

Then, the following map is an onto-homeomorphism:

$$L : \text{Hit}_n(P) \to \mathbb{R}_N; L([\text{hit}]) = (p(h AB), p(h BC), p(h CA), pqr(e T_0, v_0), pqr(e T_1, v_1)).$$

Remark. The shearing invariants of the closed leaves in boundary are determined by other invariants.
The Bonahon-Dreyer’s parametrization for our case.

We apply the Bonahon-Dreyer’s parametrization to our case.

- \( P \): a pair of pants.
- \( \mathcal{L} \): the maximal lamination.

Then, the following map is an onto-homeomorphism:

\[
\Phi_{\mathcal{L}} : \text{Hit}_n(P) \rightarrow \mathbb{R}^N, \\
\Phi_{\mathcal{L}}([\rho]) = (\sigma^\rho_P(h_{AB}), \cdots, \sigma^\rho_P(h_{BC}), \cdots, \sigma^\rho_P(h_{CA}), \cdots, \\
\tau^\rho_{pqr}(\tilde{T}_0, v_0), \cdots, \tau^\rho_{pqr}(\tilde{T}_1, v_1), \cdots).
\]

Remark. The shearing invariants of the closed leaves in boundary are determined by other invariants.
We apply the Bonahon-Dreyer’s parametrization to our case.

- $P$: a pair of pants.
- $\mathcal{L}$: the maximal lamination.

Then, the following map is an onto-homeomorphism:

\[ \Phi_{\mathcal{L}} : \text{Hit}_n(P) \rightarrow \mathbb{R}^N, \]

\[ \Phi_{\mathcal{L}}([\rho]) = (\sigma_{p}(h_{AB}), \cdots, \sigma_{p}(h_{BC}), \cdots, \sigma_{p}(h_{CA}), \cdots, \tau_{pqr}(\tilde{T}_0, v_0), \cdots, \tau_{pqr}(\tilde{T}_1, v_1), \cdots). \]

**Remark.**

The shearing invariants of the closed leaves in boundary are determined by other invariants.
Outline of the computation

**Goal:** To describe $\text{Fuch}_n(S)$ explicitly by using the Bonahon-Dreyer’s parametrization for a pair of pants. In particular, we compute $\Phi_{\mathcal{L}}([\rho_n])$ for any $n$-Fuchsian representation.
Goal: To describe $\text{Fuch}_n(S)$ explicitly by using the Bonahon-Dreyer’s parametrization for a pair of pants. In particular, we compute $\Phi_L([\rho_n])$ for any $n$-Fuchsian representation.

Outline.

1. We parameterize $\rho \in \mathcal{F}_2(P)$ by the hyperbolic length of the boundary components.
2. Describe the flag curve $\xi_{\rho_n}$ of $\rho_n = \iota_n \circ \rho$.
3. Compute the invariants $\sigma_{p}^{\rho_n}, \tau_{pqr}^{\rho_n}$. 

1. Parameterizing Fuchsian representations

- \( m \in \mathcal{T}(P) \): a hyperbolic structure on \( P \).
- \( S \): the set of simple closed curves.
- \( l_m : S \rightarrow \mathbb{R}_{>0} \): the length function associated to \( m \).
1. Parameterizing Fuchsian representations

- $\mathbf{m} \in \mathcal{T}(P)$: a hyperbolic structure on $P$.
- $\mathcal{S}$: the set of simple closed curves.
- $l_m : \mathcal{S} \to \mathbb{R}_{>0}$: the length function associated to $\mathbf{m}$.

**Proposition**

The following map is a diffeomorphism.

$$\mathcal{T}(P) \rightarrow \mathbb{R}_{>0}^3,$$

$$\mathbf{m} \leftrightarrow (l_m(A), l_m(B), l_m(C)).$$
1. Parameterizing Fuchsian representations

- \( m = (l_A, l_B, l_C) \): the hyperbolic length of the boundary components.
- \( a, b, c \): the homotopy classes of \( A, B, C \).
- \( \pi_1(P) = \langle a, b, c | abc = 1 \rangle \): a presentation.
- \( \rho \): a Fuchsian representation.
1. Parameterizing Fuchsian representations

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- \( a, b, c \): the homotopy classes of \( A, B, C \).
- \( \pi_1(P) = \langle a, b, c \mid abc = 1 \rangle \): a presentation.
- \( \rho \): a Fuchsian representation.

**Proposition (I.)**

If \( \rho \) is a Fuchsian representation associated to \( \mathbf{m} \), then \( \rho \) is conjugate to the following representation.

\[
\rho(a) = \begin{bmatrix}
\alpha & \alpha \beta \gamma + \alpha^{-1} \\
0 & \alpha^{-1}
\end{bmatrix}, \quad \rho(b) = \begin{bmatrix}
\gamma & 0 \\
-\beta^{-1} - \gamma^{-1} & \gamma^{-1}
\end{bmatrix}
\]

where \( \alpha, \beta, \gamma : \mathbb{R}_>^3 \to \mathbb{R}_> \) is defined by

\[
\alpha(l_A, l_B, l_C) = e^{l_A/2}, \quad \beta(l_A, l_B, l_C) = e^{(l_C-l_A)/2}, \quad \gamma(l_A, l_B, l_C) = e^{-l_B/2}.
\]
2. Describing the flag curve $\xi_\rho : \partial_\infty \tilde{S} \rightarrow \text{Flag} (\mathbb{R}^n)$

- $\rho_n = \iota_n \circ \rho \in \mathcal{F}_n (P)$: an $n$-Fuchsian representation.
- $\text{Dev}_\rho : \tilde{P} \rightarrow \mathbb{H}^2$: the developing map associated to $\rho$.
- The developing map $\text{Dev}_\rho$ gives the embedding $\partial_\infty P \rightarrow \partial_\infty \mathbb{H}^2$. 
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- The developing map $\text{Dev}_\rho$ gives the embedding $\partial_\infty P \to \partial_\infty \mathbb{H}^2$.
- $V = \text{Span}_\mathbb{R} \langle X^{n-1}, X^{n-2}Y, \ldots, Y^{n-1} \rangle$: an $n$-dim. $\mathbb{R}$-vec. sp.
2. Describing the flag curve $\xi_\rho : \partial_\infty \mathring{S} \to \text{Flag}(\mathbb{R}^n)$

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- The developing map $\text{Dev}_\rho$ gives the embedding $\partial_\infty P \to \partial_\infty \mathbb{H}^2$.
- $V = \text{Span}_{\mathbb{R}} < X^{n-1}, X^{n-2}Y, \ldots, Y^{n-1}>$: an $n$-dim. $\mathbb{R}$-vec. sp.

We define a subspace $W^{(i)}(z)$ of $V$ for $z \in \partial_\infty \mathbb{H}^2$ as follows.

$$W^{(i)}(z) = \text{the set of polynomials which can be divided by}$$

$$\begin{cases} 
(zX + Y)^{n-i} & (z \neq \infty) \\
X^{n-i} & (z = \infty).
\end{cases}$$

Set $W^{(0)}(z) = 0$ for any $z$. 
2. Describing the flag curve \( \xi_\rho : \partial_\infty \hat{\mathcal{S}} \to \text{Flag}(\mathbb{R}^n) \)

- \( \rho_n = \iota_n \circ \rho \in \mathcal{F}_n(P) \): an \( n \)-Fuchsian representation.
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X^{n-i} & (z = \infty).
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Set \( W^{(0)}(z) = 0 \) for any \( z \).

\[
\xi_\rho_n : \partial_\infty P \xrightarrow{\text{Dev}_\rho} \partial_\infty \mathbb{H}^2 \to \text{Flag}(\mathbb{R}^n)
\]

\[
z \mapsto (W^{(i)}(z))_{i=0,\ldots,n}
\]
Computation (1)

$$\tau_{pqr}^{\rho_n}(\tilde{T}_0, \infty)$$

Set $E = \xi_{\rho_n}(\infty)$, $F = \xi_{\rho_n}(1)$, $G = \xi_{\rho_n}(0)$. 
Set $E = \xi_{\rho_n}(\infty)$, $F = \xi_{\rho_n}(1)$, $G = \xi_{\rho_n}(0)$.

**Flags.**

\[ E^{(p)} = \text{Span}_R < X^{n-1}, X^{n-2} Y, \ldots, X^{n-p} Y^{p-1} >, \]
\[ F^{(q)} = \text{Span}_R < (X + Y)^{n-q} X^{q-1}, (X + Y)^{n-q} X^{q-2} Y, \ldots, (X + Y)^{n-q} Y^{q-1} >, \]
\[ G^{(r)} = \text{Span}_R < Y^{n-1}, XY^{n-2}, \ldots, X^{r-1} Y^{n-r} >. \]
Computation (1)

\[ \tau_{pqr}^{\rho_n}(\tilde{T}_0, \infty) \]

Set \( E = \xi_{\rho_n}(\infty) \), \( F = \xi_{\rho_n}(1) \), \( G = \xi_{\rho_n}(0) \).

**Flags.**

\[
E^{(p)} = \text{Span}_\mathbb{R} < X^{n-1}, X^{n-2}Y, \ldots, X^{n-p}Y^{p-1} >, \\
F^{(q)} = \text{Span}_\mathbb{R} < (X + Y)^{n-q}X^{q-1}, (X + Y)^{n-q}X^{q-2}Y, \ldots, (X + Y)^{n-q}Y^{q-1} >, \\
G^{(r)} = \text{Span}_\mathbb{R} < Y^{n-1}, XY^{n-2}, \ldots, X^{r-1}Y^{n-r} >. 
\]

**Nonzero elements.**

\[
e^{(p)} = X^{n-1} \wedge X^{n-2}Y \wedge \ldots \wedge X^{n-p}Y^{p-1}, \\
f^{(q)} = (X + Y)^{n-q}X^{q-1} \wedge (X + Y)^{n-q}X^{q-2}Y \wedge \ldots \wedge (X + Y)^{n-q}Y^{q-1}, \\
g^{(r)} = Y^{n-1} \wedge XY^{n-2} \wedge \ldots \wedge X^{r-1}Y^{n-r}. 
\]
\[ \tau_{pqr}^\rho (\tilde{T}_0, \infty) = \log \frac{X(p + 1, q, r - 1)}{X(p - 1, q, r + 1)} \cdot \frac{X(p, q - 1, r + 1)}{X(p, q + 1, r - 1)} \cdot \frac{X(p - 1, q + 1, r)}{X(p + 1, q - 1, r)} \]

where \( X(p, q, r) = e^p \wedge f^q \wedge g^r \).
Computation (2)

\[ \tau_{pqr}^\rho (\tilde{T}_0, \infty) = \log \frac{X(p + 1, q, r - 1)}{X(p - 1, q, r + 1)} \cdot \frac{X(p, q - 1, r + 1)}{X(p, q + 1, r - 1)} \cdot \frac{X(p - 1, q + 1, r)}{X(p + 1, q - 1, r)} \]

where \( X(p, q, r) = e(p) \land f(q) \land g(r) \).

Fix a basis \( b_1 = X^{n-1}, b_2 = X^{n-2}Y, \ldots, b_n = Y^{n-1} \) of \( V \). Then

\[
(X + Y)^{n-q} X^{q-k} Y^{k-1} = \binom{n-q}{0} b_k + \binom{n-q}{1} b_{k+1} + \cdots + \binom{n-q}{n-q} b_{n-q+k}.
\]
\[ \tau_{pqr}(\widetilde{T}_0, \infty) = \log \frac{X(p + 1, q, r - 1) \cdot X(p, q - 1, r + 1) \cdot X(p - 1, q + 1, r)}{X(p - 1, q, r + 1) \cdot X(p, q + 1, r - 1) \cdot X(p + 1, q - 1, r)} \]

where \( X(p, q, r) = e^{(p)} \wedge f^{(q)} \wedge g^{(r)} \).

Fix a basis \( b_1 = X^{n-1}, b_2 = X^{n-2}Y, \ldots, b_n = Y^{n-1} \) of \( V \). Then

\[
(X+Y)^{n-q}X^{q-k}Y^{k-1} = \binom{n-q}{0} b_k + \binom{n-q}{1} b_{k+1} + \cdots + \binom{n-q}{n-q} b_{n-q+k}.
\]

**Notation.**

\[
\binom{n}{p} = \begin{cases} 
\frac{n!}{p!(n-p)!} & (0 \leq p \leq n) \\
0 & \text{(otherwise)}
\end{cases}
\]
Computation (3)

\[ A_{pqr}(T_0) = \begin{pmatrix} (p+r) & (p+r) & \cdots & (p+r) \\ 0 & (p+r) & \cdots & (p+r) \\ (p+r) & (p+r) & \cdots & (p+r) \\ 1 & (p+r) & \cdots & (p+r) \\ \vdots & \vdots & \cdots & \vdots \\ (p+r) & (p+r) & \cdots & (p+r) \\ (p+r+1) & (p+r) & \cdots & (p+r) \end{pmatrix}, \]

\[ X(p, q, r) = \begin{pmatrix} \text{Id}_p & 0 & \text{Id}_r \\ 0 & A_{pqr}(T_0) & 0 \\ \text{Id}_r & 0 & \text{Id}_r \end{pmatrix}, \]

and \( X(p, 0, r) = 1 \) for all \( p, r \).
Result 1

Triangle invariant $\tau_{pqr}^n(\tilde{T}_0, \infty)(p, q, r \geq 1 \text{ s.t. } p + q + r = n)$.

$$\tau_{pqr}^n(\tilde{T}_0, \infty) = \log \frac{X_{T_0}(p + 1, q, r - 1)}{X_{T_0}(p - 1, q, r + 1)} \cdot \frac{X_{T_0}(p, q - 1, r + 1)}{X_{T_0}(p, q + 1, r - 1)} \cdot \frac{X_{T_0}(p - 1, q + 1, r)}{X_{T_0}(p + 1, q - 1, r)}$$

where

$$X_{T_0}(p, q, r) = \begin{vmatrix} \binom{p+r}{p} & \cdots & \binom{p+r}{p-q+1} \\ \vdots & \ddots & \vdots \\ \binom{p+r}{p+q-1} & \cdots & \binom{p+r}{p} \end{vmatrix}$$

if $q \neq 0$ and $X_{T_0}(p, 0, r) = 1$ for all $p, r$. 
Result 2

Triangle invariant $\tau_{pqr}^n(\tilde{T}_1, \infty)(p, q, r \geq 1 \text{ s.t. } p + q + r = n)$.

$$\tau_{pqr}^n(\tilde{T}_1, \infty) = \log \frac{X_{T_1}(p + 1, q, r - 1)}{X_{T_1}(p - 1, q, r + 1)} \cdot \frac{X_{T_1}(p, q - 1, r + 1)}{X_{T_1}(p + 1, q, r - 1)} \cdot \frac{X_{T_1}(p - 1, q + 1, r)}{X_{T_1}(p + q, r - 1, r)}$$

where

$$X_{T_1}(p, q, r) = (-1)^q (r+1) \begin{vmatrix} \binom{p+q}{p}(-\beta\gamma)^q & \cdots & \binom{p+q}{p-r+1}(-\beta\gamma)^{q+r-1} \\ \vdots & \ddots & \vdots \\ \binom{p+q}{p+r-1}(-\beta\gamma)^{q-r+1} & \cdots & \binom{p+q}{p}(-\beta\gamma)^r \end{vmatrix}$$

if $r \neq 0$ and $X_{T_1}(p, q, 0) = (-1)^q$ for all $p, q$. 
Result 3

Shearing invariant $\sigma_p^{\rho_n}(h_{AB})(1 \leq p \leq n - 1)$.

$$
\sigma_p^{\rho_n}(h_{AB}) = \log \frac{Y_{h_{AB}}(p)}{Y'_{h_{AB}}(p)} \cdot \frac{Y'_{h_{AB}}(p-1)}{Y_{h_{AB}}(p-1)}
$$

where

$$
Y_{h_{AB}}(p) = \binom{n-1}{p} (\beta \gamma)^{n-p-1},
$$

$$
Y'_{h_{AB}}(p) = (-1)^{n-p-1} \binom{n-1}{p}.
$$
Result 4

Shearing invariant $\sigma_p^{\rho_n}(h_{BC})(1 \leq p \leq n - 1)$.

$$\sigma_p^{\rho_n}(h_{BC}) = \log \frac{Y_{h_{BC}}(p)}{Y'_{h_{BC}}(p)} \cdot \frac{Y'_{h_{BC}}(p - 1)}{Y_{h_{BC}}(p - 1)}$$

where

$$Y_{h_{BC}}(p) = (-1)^{(n-p)p} \begin{vmatrix} (p+1) & \cdots & (-p+1) & (n-1)(\frac{\beta}{\beta + \gamma})^{n-1} \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ (p+1) & \cdots & (p+1) & (n-1)(\frac{\beta}{\beta + \gamma})^{p} \end{vmatrix}$$

if $p \neq n - 1$ and $Y_{h_{BC}}(n - 1) = (-1)^{n-1}(\frac{\beta}{\beta + \gamma})^{n-1}$,

$$Y'_{h_{BC}}(p) = (-1)^{np+n+1} \begin{vmatrix} (p+1) & \cdots & (-p+1) \\ 1 & \cdots & 1 \\ \vdots & \vdots & \vdots \\ (p+1) & \cdots & (p+1) \end{vmatrix}$$

if $p \neq n - 1$ and $Y'_{h_{BC}}(n - 1) = (-1)^{n-1}$. 
Result 5

Shearing invariant $\sigma_p^{\rho_n}(h_{CA})(1 \leq p \leq n - 1)$.

$$\sigma_p^{\rho_n}(h_{CA}) = \log - \frac{Y_{h_{CA}}(p)}{Y'_{h_{CA}}(p)} \cdot \frac{Y'_{h_{CA}}(p - 1)}{Y_{h_{CA}}(p - 1)}$$

where

$$Y_{h_{CA}}(p) = (-1)^{np} \begin{vmatrix} \binom{n-p}{n-p-1} & \cdots & \binom{n-p}{n-2p} & \binom{n-1}{n-p-1}(\alpha^2 \beta \gamma + 1)^p \\ \vdots & \ddots & \vdots & \vdots \\ \binom{n-p}{n-1} & \cdots & \binom{n-p}{n-p} & \binom{n-1}{n-1}(\alpha^2 \beta \gamma + 1)^0 \end{vmatrix}$$

if $p \neq 0$ and $Y_{h_{CA}}(0) = 1$,

$$Y'_{h_{CA}}(p) = (-1)^{np} \begin{vmatrix} \binom{n-p}{n-p-1} & \cdots & \binom{n-p}{n-2p} \\ \vdots & \ddots & \vdots \\ \binom{n-p}{n-2} & \cdots & \binom{n-p}{n-p-1} \end{vmatrix}$$

if $p \neq 0$ and $Y'_{h_{CA}}(0) = 1$. 
Thank you for your attention!!