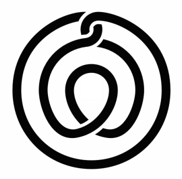
On adequacy
and the crossing number of satellite knots



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20th October, 2017

Agenda

Preliminaries

$\downarrow$

Link adequacy

$\downarrow$

Link parallels

$\downarrow$

Cable knots

$\downarrow$

Main result

$\downarrow$

Summary

Definition (*Satellite knot*)

P: knot in ST. (*Pattern*)
C: knot in $\mathbb{S}^3$ with framing 0. (*Companion*)
$e : ST \hookrightarrow N(C)$: faithful embedding.
Then eP is called a *satellite knot* (of C). From here on $eP =: Sat(P, C)$.

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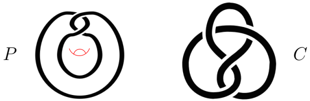

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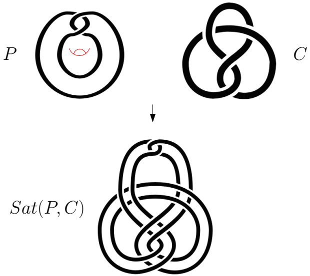

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GOAL

Problem 1.65 (*Kirby, 1995*)

Is the crossing number $cr(K)$ of a knot K additive with respect to connected sum, that is, is the equality $cr(K_1 \# K_2) = cr(K_1) + cr(K_2)$ true?

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- $cr(K_1 \# ... \# K_n) \geq \frac{cr(K_1)+...+cr(K_n)}{152}$. *(Lackenby, 2011)*

Link adequacy

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A *state* of a link is a function

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The *Kauffman bracket* of a link with diagram D can be written as:

$$\langle D \rangle = \sum_s \left(A^{\sum_{i=1}^n s(i)} (-A^{-2} - A^2)^{|sD|-1} \right).$$

- s_+ is the state for which $\sum_{i=1}^{n} s_+(i) = n$
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D is *plus-adequate* if $|s_+ D| > |sD|$ for all s with $\sum_{i=1}^{n} s(i) = n - 2$.
D is *minus-adequate* if $|s_- D| > |sD|$ for all s with $\sum_{i=1}^{n} s(i) = -n + 2$.
D is *adequate* if *plus-adequate* and *minus-adequate*.

Lemma 1 (*Lickorish*)

Let D be a link diagram with n crossings.

1. $M_{\langle D \rangle} \leq n + 2|s_+ D| - 2$, with equality if D is plus-adequate,
2. $m_{\langle D \rangle} \geq -n - 2|s_- D| + 2$, with equality if D is minus-adequate.

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Corollary 1 (*Lickorish*)

If D is adequate:

$$B(\langle D\rangle) = M_{\langle D\rangle} - m_{\langle D\rangle} = 2n + 2|s_+D| + 2|s_-D| - 4.$$

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Lemma 2 (*Lickorish*)

Let D be a connected link diagram with n crossings.

$$|s_+D| + |s_-D| \leq n + 2,$$

with equality if D alternating.

Lemma 3

Let D be a diagram of an oriented link L.

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Theorem 1 (*Lickorish*)

Let D be a connected, n-crossing diagram of an oriented link L.

1. $B(J(L)) \leq n$,
2. if D is alternating and reduced, $B(J(L)) = n$.

Link parallels

Definition

Let D be a diagram of an oriented link L. The r-parallel of D is the same diagram where each link component has been replaced by r parallel copies of it, all preserving their "over" and "under" strands as in the original diagram.

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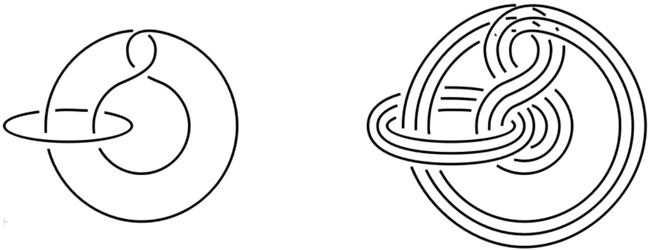

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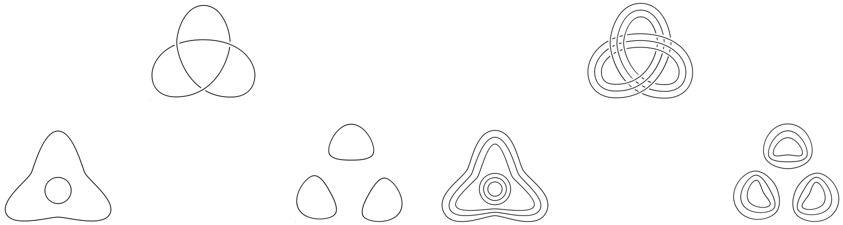

We construct the parallel versions of the given results.

Lemma 1-*bis* (*JP*)

Let D be a link diagram with n crossings.

1. $M_{\langle D^r \rangle} \leq nr^2 + 2r|s_+D| - 2$, with equality if D is plus-adequate,
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Corollary 1-*bis*

If D is adequate:

$$B(\langle D^r \rangle) = M_{\langle D^r \rangle} - m_{\langle D^r \rangle} = 2nr^2 + 2r|s_+D| + 2r|s_-D| - 4.$$

Theorem 2 (*JP*)

Let L be an adequate oriented link.

$$cr(L^r) \geq \frac{r^2}{2} cr(L) + 2r - 1.$$

Proof follows from *Theorem 1*, *Lemma 3* and *Lemma 1 (bis)*.

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Corollary 3

If L is alternating:

$$cr(L^r) \geq \frac{r(r+1)}{2} cr(L) + r - 1.$$

Proof uses *Theorem 2* and *Lemma 2*.

Cable knots

Definition

D: diagram of $C \subset \mathbb{S}^3$.
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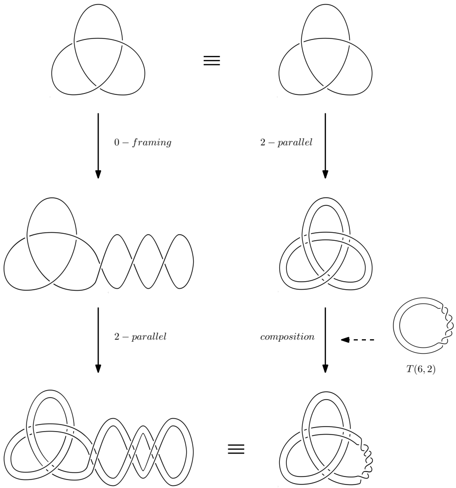

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Hardest result to prove.
Uses all previous lemmata and theorems.

Theorem 3 (*JP*, 2015)

Let $P \subset ST$ such that $J_{ST}(P) = \sum_{k=0}^{M} \beta_k z_{ST}^k$ with $\beta_M \neq 0$, let $C \subset \mathbb{S}^3$, and let $Sat(P, C)$ be their satellite knot.

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Corollary 4

In particular,

$$B(J(Sat(P, C))) \geq B(J(C; k)).$$

Main result

Theorem (*JP*)

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$$cr(Sat(P, C)) \geq B(J(Sat(P, C))) \geq B(J(C; M)) \geq B(J(C^M)) \geq M \cdot cr(C).$$

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Summary

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- It is true for adequate knots.

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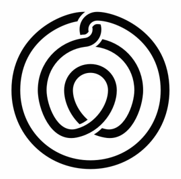

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