Parametrization for intersecting 3-punctured spheres in hyperbolic 3-manifolds

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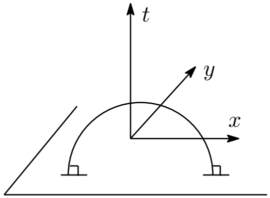
The hyperbolic space



$\mathbb{H}^3$: the hyperbolic 3-space
$\cong \{(x, y, t) \in \mathbb{R}^3 | t > 0\}$
with the metric $ds^2 = (dx^2 + dy^2 + dt^2)/t^2$
(the upper half-space model)

geodesic in $\mathbb{H}^3$ —
circular arc or line orthogonal to the plane $\{(x, y, 0) \in \mathbb{R}^3\}$

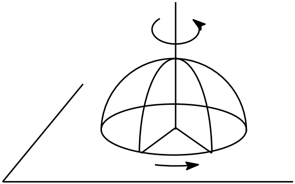
Identifications:

- $\{(x, y, 0) \in \mathbb{R}^3\} \cong \mathbb{C}$ $(\ni x + iy)$
- $\partial\mathbb{H}^3 \cong \mathbb{C} \cup \{\infty\}$
- ori.-preserving isometry of $\mathbb{H}^3 \longleftrightarrow$ Möbius transformation of $\mathbb{C} \cup \{\infty\}$
  $\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2, \mathbb{C})$ $(= \mathrm{SL}(2, \mathbb{C})/\pm 1)$
- $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}(2, \mathbb{C}) \cong \{z = x + iy \mapsto \dfrac{az + b}{cz + d}\}$

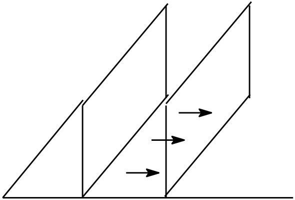
Types of isometries of $\mathbb{H}^3$

An orientation-preserving isometry of $\mathbb{H}^3$ is one of the following types:

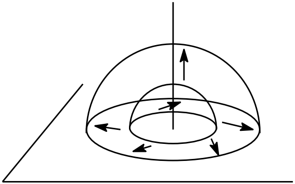
- identity
- elliptic — fixing pointwise a geodesic in $\mathbb{H}^3$
- parabolic — fixing a single point in $\partial\mathbb{H}^3$
- hyperbolic (loxodromic) — fixing two points in $\partial\mathbb{H}^3$ (fixing setwise a geodesic in $\mathbb{H}^3$)



elliptic



parabolic



hyperbolic

Types and traces

The types are determined by the trace.

- elliptic $\sim \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \in \mathrm{PSL}(2,\mathbb{C})$ $(|\lambda| = 1) \iff -2 < \mathrm{trace} < 2$
- parabolic $\sim \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \mathrm{PSL}(2,\mathbb{C}) \iff \mathrm{trace} = \pm 2$
- hyperbolic $\sim \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \in \mathrm{PSL}(2,\mathbb{C})$ $(|\lambda| \neq 1) \iff \mathrm{trace} \notin [-2, 2]$

(up to conjugacy)

Hyperbolic 3-manifold

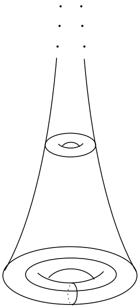
M: an orientable hyperbolic 3-manifold
(hyperbolic $:\Longleftrightarrow$ having a complete metric of sectional curvature -1)
$\Longrightarrow M \cong \mathbb{H}^3/\pi_1(M)$, where $\pi_1(M)$ is regarded as a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$.

- elliptic element $\notin \pi_1(M)$
- parabolic element $\in \pi_1(M) \longleftrightarrow$ loop in a "cusp" of M
- hyperbolic element $\in \pi_1(M) \longleftrightarrow$ closed geodesic in M

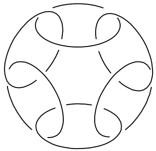
(up to conjugacy)

Cusp of a hyperbolic 3-manifold

If $\mathrm{vol}(M) < \infty$, then M is homeomorphic to the interior of a compact 3-manifold $\overline{M}$ with boundary consisting tori. In this case, a cusp of M is a component of $\partial \overline{M}$. (In general, there may be annular cusps in the boundary.)
A neighborhood of a torus cusp is isometric to $\{(x, y, t) \in \mathbb{H}^3 | t > t_0\}/\langle z \mapsto z + 1, z \mapsto z + \tau \rangle$ for some $t_0 > 0$ and $\tau \in \mathbb{C}$ with $\mathrm{Im}(\tau) > 0$. ($z = x + iy$.) τ is called the modulus of the cusp T with respect to fixed generators of $\pi_1(T)$.



There are many links whose complement is hyperbolic. For example:



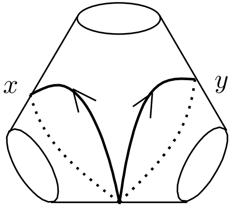
3-punctured sphere in a hyperbolic 3-manifold

A 3-punctured sphere (a.k.a. a pair of pants) is an orientable surface of genus 0 with 3 punctures.
A totally geodesic 3-punctured sphere is the double of an ideal hyperbolic triangle. The (complete) hyperbolic structure of a 3-punctured sphere is unique.

Theorem (Adams (1985))

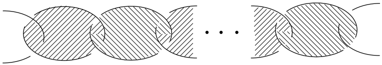
An essential (properly embedded) 3-punctured sphere in a hyperbolic 3-manifold is isotopic to a totally geodesic 3-punctured sphere.

By taking conjugacies,
we may assume $x = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $y = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$.
$\mathrm{tr}(xy^{-1}) = \pm 2 \iff c = 0$ or 2.
Then we have $c = 2$.

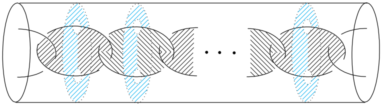


Union of 3-punctured spheres

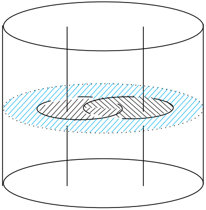
$A_n \quad (n \geq 1)$



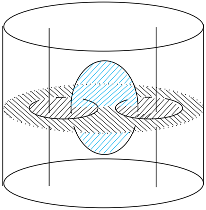
$B_{2n} \quad (n \geq 1)$



T_3



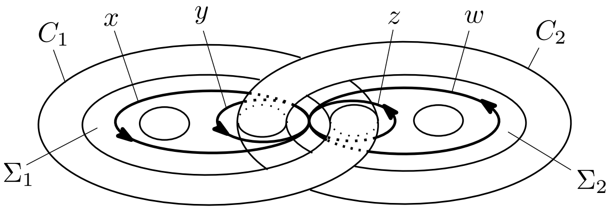
T_4



...etc.

The index indicates the number of 3-punctured spheres.

The metric of neighborhood of 3-punctured spheres of the type A_n for $n \geq 2$ is determined by the modulus $\tau \in \mathbb{C}$ of cusps. (The moduli of the adjacent cusps coincide.)



We may assume that

$$x = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, y = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, z = \begin{bmatrix} 1 & 2/\tau \\ 0 & 1 \end{bmatrix}, w = \begin{bmatrix} 1 & 0 \\ 2\tau & 1 \end{bmatrix} \in \mathrm{PSL}(2, \mathbb{C}).$$

($xy^{-1} = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$ and $zw^{-1} = \begin{bmatrix} -3 & 2/\tau \\ -2\tau & 1 \end{bmatrix}$ are parabolic.)

Bounds of moduli

Consider the set $\mathcal{C}_n := \{\tau \in \mathbb{C} | \tau$ is the modulus for 3-punctured spheres of A_n contained in a hyperbolic 3-manifold$\}$.
(not assumed to have finite volume)

Lemma (The Shimizu-Leutbecher lemma)

Suppose that a group generated by two elements $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}(2, \mathbb{C})$ *is discrete. Then* $c = 0$ *or* $|c| \geq 1/2$.

We apply the Shimizu-Leutbecher lemma for

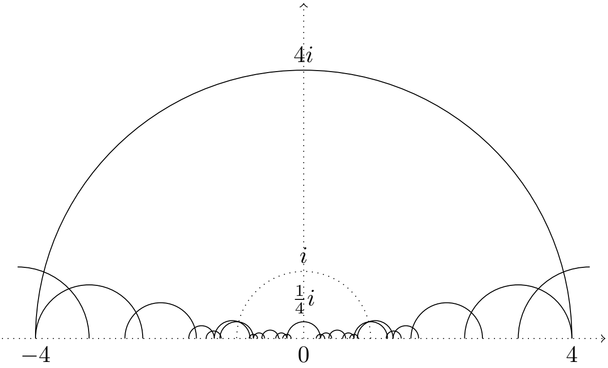
$$\langle x = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, y^n w^m = \begin{bmatrix} 1 & 0 \\ 2m\tau + 2n & 1 \end{bmatrix} \rangle \text{ and}$$

$$\langle y = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, x^n z^m = \begin{bmatrix} 1 & (2m/\tau) + 2n \\ 0 & 1 \end{bmatrix} \rangle.$$

Rough bound

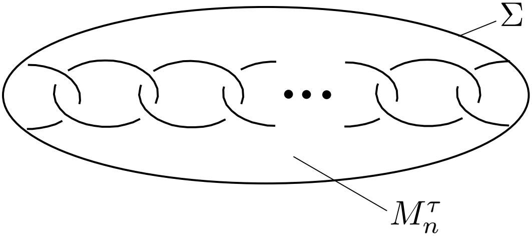
Proposition

$\tau \in \mathcal{C}_2$ satisfies the following inequalities:
$|m\tau + n| \geq 1/4$ and $|(m/\tau) + n| \geq 1/4$ for $(m, n) \in \mathbb{Z} \times \mathbb{Z} \setminus (0, 0)$.
In particular, $\frac{1}{4} \leq |\tau| \leq 4$ and $0.079 \leq \arg \tau \leq \pi - 0.079$.



Sets of moduli

Consider the 4-punctured sphere Σ near the 3-punctured spheres of A_n in a hyperbolic 3-manifold M. (Σ bounds M_n^τ.)



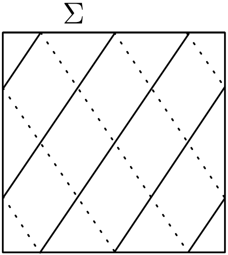
$\mathcal{C}_n^{\text{incomp}} := \{\tau \in \mathcal{C}_n | \Sigma \text{ is incompressible}\}$
$\mathcal{C}_n^{\text{comp}} := \{\tau \in \mathcal{C}_n | \Sigma \text{ is compressible}\}$
$\mathcal{C}_n = \mathcal{C}_n^{\text{incomp}} \cup \mathcal{C}_n^{\text{comp}}$

The case that Σ is incompressible

$\tau \in \mathcal{C}_n^{\mathrm{incomp}} \Longrightarrow M_n^\tau$ extends an infinite volume hyperbolic 3-manifold homeomorphic to M_n^τ.
$\mathrm{int}(\mathcal{C}_n^{\mathrm{incomp}}) \cong$ the Teichmüller space of Σ (homeomorphic to an open disk)
$\mathcal{C}_n^{\mathrm{incomp}} = \mathrm{cl}(\mathrm{int}(\mathcal{C}_n^{\mathrm{incomp}}))$ (homeomorphic to a closed disk)
$\partial \mathcal{C}_n^{\mathrm{incomp}} \cong \mathbb{R} \cup \{\infty\}$

- rational point in $\mathbb{R}$ or $\infty \longleftrightarrow$ cusp
- irrational point in $\mathbb{R} \longleftrightarrow$ ending lamination

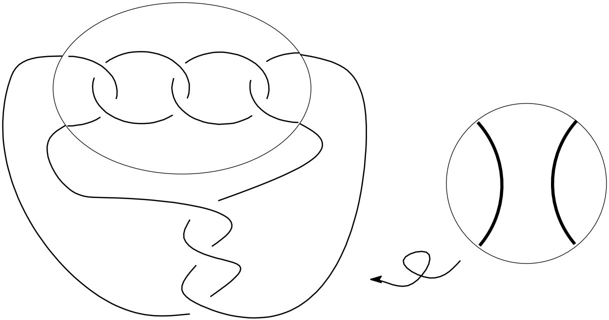
Example: slope $= 3/2$



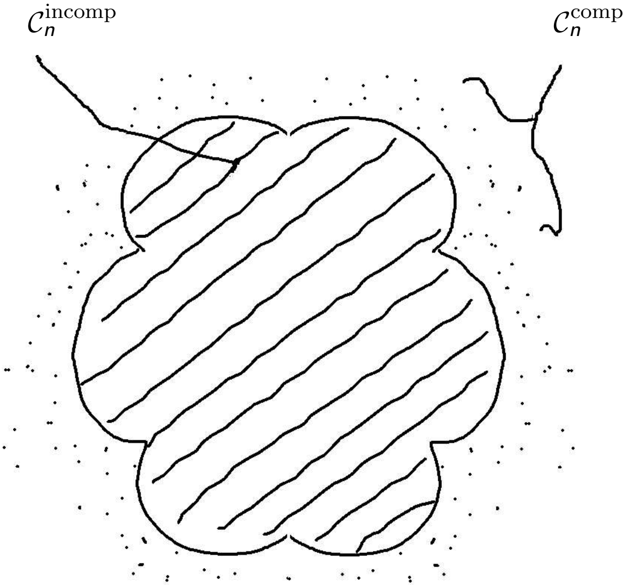
The case that Σ is compressible

$\tau \in \mathcal{C}_n^{\text{comp}} \Longrightarrow M = M_n^{\tau} \cup$ (a trivial tangle).

- $\mathcal{C}_2^{\text{comp}} \longleftrightarrow \mathbb{Q} \setminus \{0, 1, 2\}$
- $\mathcal{C}_3^{\text{comp}} \longleftrightarrow \mathbb{Q} \setminus \{0\}$
- $\mathcal{C}_n^{\text{comp}} \longleftrightarrow \mathbb{Q}$ $(n \geq 4)$



Shape of $\mathcal{C}_n$



For hyperbolic 3-manifolds of finite volume

$\mathcal{C}_n^{\mathrm{fin}} := \{\tau \in \mathbb{C} | \tau$ is the modulus for 3-punctured spheres of A_n contained in a hyperbolic 3-manifold of *finite volume*$\}$
$\mathcal{C}_n^{\mathrm{comp}} \subset \ \mathcal{C}_n^{\mathrm{fin}}$.

Theorem (Brooks (1986))

Let $\Gamma < \mathrm{PSL}(2, \mathbb{C})$ be a geometrically finite Kleinian group. Then there exist arbitrarily small quasi-conformal deformations Γ_ϵ of Γ, such that Γ_ϵ admits an extension of the fundamental group of a finite volume hyperbolic 3-manifold.

Corollary

$\mathcal{C}_n^{\mathrm{fin}}$ is dense in $\mathcal{C}_n$.

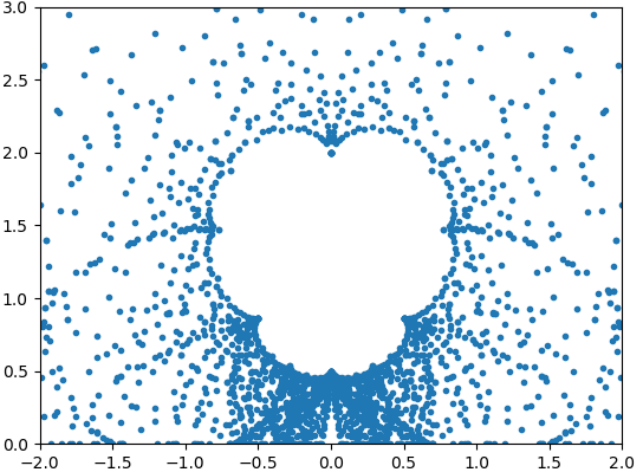
Way to compute

To plot points of $\partial\mathcal{C}_n^{\mathrm{incomp}}$, consider the condition that a simple loop in Σ is an annular cusp.
Solve equations:
$-2 =$ trace of an element represented by a simple loop in Σ.
We will not avoid plotting unnecessary points outside $\mathcal{C}_n^{\mathrm{incomp}}$.

Remark: $\mathcal{C}_{n+1} \subset \mathcal{C}_n$, $\mathcal{C}_{n+1}^{\mathrm{incomp}} \subset \mathcal{C}_n^{\mathrm{incomp}}$, $\mathcal{C}_{n+2} \subset \mathcal{C}_n^{\mathrm{incomp}}$.

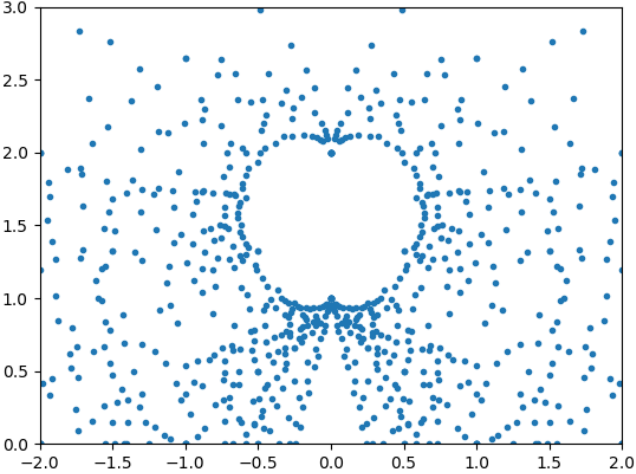
Computation for $\mathcal{C}_2^{\mathrm{incomp}}$

Caution: No information of $\mathcal{C}_2^{\mathrm{comp}}$.



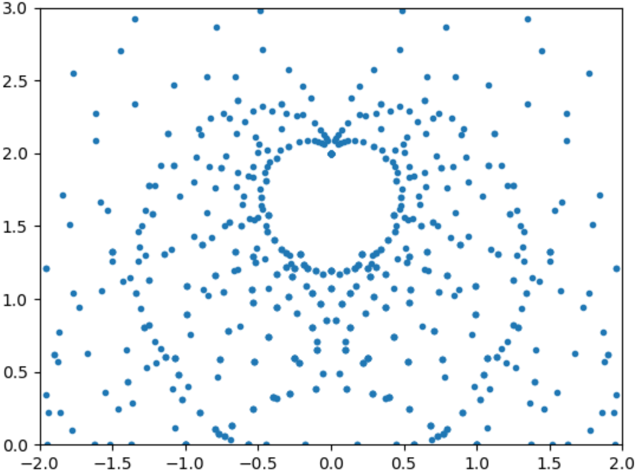
Computation for $\mathcal{C}_3^{\mathrm{incomp}}$

Caution: No information of $\mathcal{C}_3^{\mathrm{comp}}$.



Computation for $\mathcal{C}_4^{\mathrm{incomp}}$

Caution: No information of $\mathcal{C}_4^{\mathrm{comp}}$.

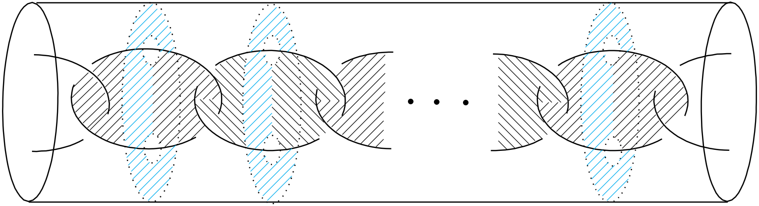


Limit of moduli

Theorem (Y.)

Let $\tau_n \in \mathcal{C}_n$ for $n \geq 2$. Then $\lim_{n\to\infty} \tau_n = 2i$.

$2i =$ the modulus for the 3-punctured spheres of A_n contained in the ones of B_{2n}.



Drilling theorem

Theorem (Brock-Bromberg (2004) + Hodgson-Kerckhoff (2008))

For any $K > 1$, there is a constant L satisfying the following condition: Let M be a finite volume hyperbolic 3-manifold. Let M_0 be a Dehn filling of M along a slope whose normalized length is more than L. Then thick parts of M and M_0 are K-bilipschitz.

The normalized length of a slope is measured after rescaling the metric on the cusp torus to have unit area.

Lemma

Suppose that a finite volume hyperbolic 3-manifold M has 3-punctured spheres $\Sigma_1, \ldots, \Sigma_n$ of the type A_n. Then M is obtained by a Dehn filling of a hyperbolic 3-manifold M_0 with 3-punctured spheres of the type B_{2n} containing $\Sigma_1, \ldots, \Sigma_n$. Moreover, the normalized length of the slope for this Dehn filling is at least $\sqrt{n+1}/2$.