Finding Roots of Any Polynomial by Random Relaxed Newton's Methods

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In this talk, we develop the theory of random holomorphic dynamics. Applying it to finding roots of polynomials by random relaxed Newton's methods, we show that for any polynomial g, for any initial value $z \in \mathbb{C}$ which is not a root of g', **the random orbit starting with z tends to a root of g almost surely**, which is **the virtue of the effect of the randomness**.

Definition 1. We use the following notations.

(1) We denote by $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \cong S^2$ the **Riemann sphere** endowed with the spherical distance.

(2) We set $\mathcal{P} := \{h : \mathbb{C} \to \mathbb{C} \mid h$ is a polynomial, $\deg(h) \geq 2\}$.

(3) We set $\Lambda := \{\lambda \in \mathbb{C} \mid |\lambda - 1| < 1\}$. Also, let $\mathfrak{M}_{1,c}(\Lambda)$ be the space of all Borel probability measures τ on Λ whose topological support supp τ is a compact subset of Λ.
For each $\tau \in \mathfrak{M}_{1,c}(\Lambda)$, let $\tilde{\tau} := \otimes_{n=1}^{\infty} \tau$. This is a Borel probability measure on $\Lambda^{\mathbb{N}}$.

(4) For each $g \in \mathcal{P}$ and for each $\lambda \in \Lambda$, let $N_{g,\lambda} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the rational map defined by $N_{g,\lambda}(z) = z - \lambda \frac{g(z)}{g'(z)}$ for each $z \in \hat{\mathbb{C}}$. This is called the **random relaxed Newton's method map** for g.

(5) For each $g \in \mathcal{P}$, let $Q_g := \{x \in \mathbb{C} \mid g(x) = 0\}$ and $\Omega_g := \mathbb{C} \setminus \{z_0 \in \mathbb{C} \mid g'(z_0) = 0, g(z_0) \neq 0\}$.

Remark 2. Note that $\sharp(\mathbb{C} \setminus \Omega_g) \leq \deg(g) - 1$.

Theorem 3 (Main Theorem). *Let $g \in \mathcal{P}$. Let $\tau \in \mathfrak{M}_{1,c}(\Lambda)$ such that*

$$\mathrm{int}(\mathrm{supp}\,\tau) \supset \{\lambda \in \mathbb{C} \mid |\lambda - 1| \leq \frac{1}{2}\},$$

where $\mathrm{int}(\mathrm{supp}\,\tau)$ *denotes the set of interior points of* $\mathrm{supp}\,\tau$ *with respect to the topology in Λ. Suppose that τ is absolutely continuous with respect to the Lebesgue measure on Λ. (e.g. suppose that τ is the normalized 2-dimensional Lebesgue measure on $\{\lambda \in \mathbb{C} \mid |\lambda - 1| \leq r\}$ where $\frac{1}{2} < r < 1$.) Then we have the following.*

- *For each $z \in \Omega_g$, there exists a Borel subset $C_{g,\tau,z}$ of $\Lambda^{\mathbb{N}}$ with $\tilde{\tau}(C_{g,\tau,z}) = 1$ satisfying that for each $\gamma = (\gamma_1, \gamma_2, \ldots) \in C_{g,\tau,z}$, there exists an element $x = x(g, \tau, z, \gamma) \in Q_g$ such that*

$$N_{g,\gamma_n} \circ \cdots \circ N_{g,\gamma_1}(z) \to x \quad \text{as} \quad n \to \infty \text{ (exponentially fast)}.$$

We say that a non-constant polynomial g is **normalized** if

$$\{z_0 \in \mathbb{C} \mid g(z_0) = 0\} \subset \mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}.$$

For a given polynomial g, sometimes it is not difficult for us to find an element $a \in \mathbb{R} \setminus \{0\}$ such that $g(az)$ is a normalized polynomial of z.

It is well-known that if $g \in \mathcal{P}$ is a normalized polynomial, then so is g'. Thus, we obtain the following corollary.

Corollary 4. *Let $g \in \mathcal{P}$ be a normalized polynomial. Let $\tau \in \mathfrak{M}_{1,c}(\Lambda)$ such that* $\mathrm{int}(\mathrm{supp}\,\tau) \supset \{\lambda \in \mathbb{C} \mid |\lambda - 1| \leq \frac{1}{2}\}$*. Suppose that τ is absolutely continuous with respect to the Lebesgue measure on Λ.*
Let $z_0 \in \mathbb{C} \setminus \mathbb{D}$.
Then for $\tilde{\tau}$-a.e. $\gamma = (\gamma_1, \gamma_2, \ldots) \in \Lambda^{\mathbb{N}}$, the orbit $\{N_{g,\gamma_n} \circ \cdots \circ N_{g,\gamma_1}(z_0)\}_{n=1}^{\infty}$ tends to a root $x = x(g, \tau, \gamma)$ of g.

Moreover, if, in addition to the assumptions of our corollary, we know the coefficients of g explicitly, then by the following algorithm (1)(2)(3) in which we consider d-random orbits of z_0 under d-different random iterations of $\{N_{g_1,\lambda}\}_\lambda, \dots, \{N_{g_d,\lambda}\}_\lambda$ for some polynomials $g_1, \dots, g_d$, we can find all roots of g almost surely with arbitrarily small errors.

(1) *Let $g_1 = g$. By Theorem 3, for $\tilde{\tau}$-a.e. $\gamma = (\gamma_1, \gamma_2, \dots) \in \Lambda^{\mathbb{N}}$, the orbit $\{N_{g_1,\gamma_n} \circ \cdots \circ N_{g_1,\gamma_1}(z_0)\}_{n=1}^{\infty}$ tends to a root $x = x(g_1, \tau, \gamma)$ of $g_1 = g$. Let x_1 be one of such $x(g, \tau, \gamma)$ (with aribitrarily small error).*

(2) *Let $g_2(z) = g_1(z)/(z - x_1)$. By using synthetic devision, we regard g_2 as a polynomial which devides g_1 (with arbitrarily small error). Note that g_2 is a normalized polynomial. By Theorem 3, for $\tilde{\tau}$-a.e.$\gamma = (\gamma_1, \gamma_2 \dots) \in \Lambda^{\mathbb{N}}$, the orbit $\{N_{g_2,\gamma_n} \circ \cdots \circ N_{g_2,\gamma_1}(z_0)\}_{n=1}^{\infty}$ tends to a root $x = x(g_2, \tau, \gamma)$ of g_2, which is also a root of g (with arbitrarily small error).*
Let x_2 be one of such $x(g_2, \tau, \gamma)$ (with arbitrarily small error).

(3) *Let $g_3(z) = g_2(z)/(z - x_2)$ and as in the above, we find a root x_3 of g_3, which is also a root of g (with arbitrarily small error).*
Continue this method.

Remark 5. **(I)** M. Hurley showed that for any $k \geq 3$, there exists a non-empty open subset A_k of $\mathcal{P}_k := \{g \in \mathcal{P} \mid \deg(g) = k\}$ such that for each $g \in A_k$, there exists a non-empty open subset U_g of $\hat{\mathbb{C}}$ such that for any $z \in U_g$, the orbit $\{N_{g,1}^n(z)\}_{n=1}^{\infty}$ cannot converge to any root of g.

(II) C. McMullen showed (Ann. of Math., 1987) that for any $k \in \mathbb{N}, k \geq 4$ and for any $l \in \mathbb{N}$, there exists **NO** rational map $\tilde{N} : \mathcal{P}_k \to \text{Rat}_l := \{f \in \text{Rat} \mid \deg(f) = l\}$ such that "for any g in an open dense subset of $\mathcal{P}_k$, for any z in an open dense subset of $\hat{\mathbb{C}}$, $\tilde{N}(g)^n(z)$ tends to a root of g as $n \to \infty$."

(III) Thus Theorem 3 and Corollary 4 deal with **randomness-induced phenomena** (new phenomena in random dynamics which cannot hold in deterministic dynamics).

Remarks.

In Theorem 3 and Corollary 4, **the size of the noise is big which enables the system to make the minimal set with period greater than 1 collapse**.

However, since the size of the noise is big, we have to develop the theory of random holomorphic dynamical systems with noise or randomness of **any size**.

Note also that we need to deal with the random systems whose **"kernel Julia sets" are not empty**.

Outline of the Proof of Theorem 3.

(1) Let g, τ be as in the assumptions of Theorem 3.
Let
$$G_\tau := \{N_{g,\gamma_n} \circ \cdots \circ N_{g,\gamma_1} \mid n \in \mathbb{N}, \gamma_j \in \text{supp}\, \tau \ (\forall j)\}.$$
This is a semigroup whose product is the composition of maps.
Let
$$F(G_\tau) := \{z \in \hat{\mathbb{C}} \mid G_\tau \text{ is equicontinuous in a nbd of } z\}.$$
This is called the **Fatou set** of G_τ.
Let
$$J(G_\tau) := \hat{\mathbb{C}} \setminus F(G_\tau).$$
This is called the **Julia set** of G_τ.
Let
$$J_{\ker}(G_\tau) := \bigcap_{h \in G_\tau} h^{-1}(J(G_\tau)).$$
This is called the **kernel Julia set** of G_τ.

(2) Montel's theorem implies that

$$\sharp(J_{\ker}(G_\tau)) < \infty.$$

(**Remark.** This cannot hold for any deterministic complex dyn. system of an f with $\deg(f) \geq 2$.)

(3) By using this and some technical arguments, we can show that there exist finitely many **attracting minimal sets** $K_1, \ldots, K_m$ of G_τ s. t. for $\forall z \in \Omega_g$, for $\tilde{\tau}$-a.e. $\gamma = (\gamma_1, \gamma_2, \ldots,) \in \Lambda^{\mathbb{N}}$, we have

$$d(N_{g,\gamma_n} \circ \cdots \circ N_{g,\gamma_1}(z), \cup_{j=1}^m K_j) \to 0 \text{ as } n \to \infty. \quad (\bigstar)$$

Here, we say that a non-empty compact subset K of $\hat{\mathbb{C}}$ is a **minimal set** of G_τ if for any $z \in K$, $\overline{\cup_{h \in G_\tau} \{h(z)\}} = K$.

Remark. Thus the chaoticity is much less than that of deterministic complex dynamical systems.

In the proof of ($\bigstar$), the **difficulty** is that $\infty \in \hat{\mathbb{C}}$ is a common repelling fixed point of $N_{g,\lambda}$ and $\infty \in J_{\ker}(G_\tau)$, thus $J_{\ker}(G_\tau) \neq \emptyset$.

(4) We can show that for each $j = 1, \ldots, m$, either
(i) $K_j = \{x\}$ for some $x \in Q_g$, or (ii) $K_j \cap Q_g = \emptyset$.

(5) We want to exclude the case (ii) $K_j \cap Q_g = \emptyset$
(if we can do that, then the proof of Theorem 3 is complete).

(6) In order to do so, let $j \in \{1, \ldots, m\}$ and suppose $K_j \cap Q_g = \emptyset$.
Then K_j contains an attracting periodic cycle $z_1, \ldots, z_p$ of $N_{g,1}$ with **period** $p \geq 2$, where $z_{i+1} = N_{g,1}(z_i)$ $(i = 1, \ldots, p)$, $z_{p+1} = z_1$.

(7) We may suppose that $|z_1 - z_2| = \max\{|z_i - z_{i+1}| \mid i = 1, \ldots, p\}$.

(8) Since $\mathrm{int}(\mathrm{supp}\,\tau) \supset \overline{D(1, \frac{1}{2})}$ by our assumption, we see

$$\mathrm{int}(\{N_{g,\lambda}(z_i) \mid \lambda \in \mathrm{supp}\,\tau\}) \supset \overline{D(z_{i+1}, \tfrac{|z_i - z_{i+1}|}{2})} \text{ for each } i = 1, 2.$$

(9) Since $\frac{|z_1 - z_2|}{2} + \frac{|z_2 - z_3|}{2} \geq |z_2 - z_3|$, it follows that

$$\mathrm{int}(K_j) \supset \mathrm{int}(\{N_{g,\lambda}(z_i) \mid \lambda \in \mathrm{supp}\,\tau, i = 1, 2\}) \supset \text{ segment } \overline{z_2 z_3},$$

which implies that $K_j \cap J(N_{g,1}) \neq \emptyset$. However, this contradicts that K_j is **attracting** and $K_j \subset F(G_\tau) \subset F(N_{g,1})$. QED.

For the preprint, see [3].

Reference:

[1] H. Sumi, *Random complex dynamics and semigroups of holomorphic maps*, Proc. London Math. Soc. (2011) 102(1), pp 50–112.

[2] H. Sumi, *Cooperation principle, stability and bifurcation in random complex dynamics*, Adv. Math., 245 (2013) pp 137–181.

[3] H. Sumi, *Negativity of Lyapunov Exponents and Convergence of Generic Random Polynomial Dynamical Systems and Random Relaxed Newton's Methods*, preprint, https://arxiv.org/abs/1608.05230. **Including the contents of this talk. 60 pages.**