Homology groups of neighborhood complexes of graphs (Introduction to topological combinatorics)

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Chromatic number

$G = (V, E)$: simple graph(no loop, no multi edge)

$[n] := \{1, 2, \ldots, n\}$

$c \colon V \to [n]$ is a coloring $\overset{def.}{\Longleftrightarrow}$ "$\{u, v\} \in E \Rightarrow c(u) \neq c(v)$"

We define the chromatic number $\chi(G)$ of G by

$$\chi(G) = \min\{n \in \mathbb{N} \mid \exists c \colon V \to [n] \text{ coloring}\}$$

We define the complete graph K_n by

$$V(K_n) = [n],\ E(K_n) = \{\{u, v\} | u, v \in [n], u \neq v\}.$$

We can easily see $\chi(K_n) = n$.

Let C_n be a graph such that

$$V(C_n) = [n],\ E(C_n) = \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}, \{n, 1\}\}.$$

n is even $\Rightarrow \chi(C_n) = 2$. n is odd $\Rightarrow \chi(C_n) = 3$.

Neighborhood complexes

$G = (V, E)$: simple graph (no loop, no multi edge)

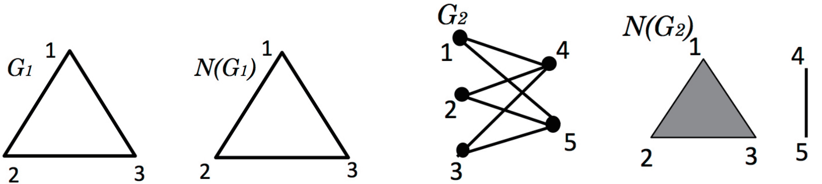
$A \subset V$

$CN(A) := \{v \mid \{v, a\} \in E \text{ for } \forall a \in A\}$

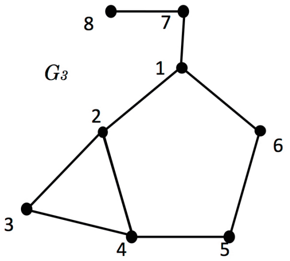
$N(G) := \{A \subset V \mid CN(A) \neq \emptyset\}$ (neighborhood complex)

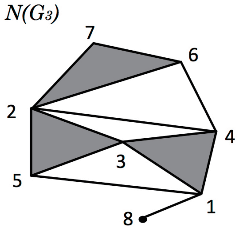
$N(G)$ is a simplicial complex.

$\{A \mid A \text{ is a maximal face of } N(G)\} \subset \{CN(\{v\}) \mid v \in V(G)\}$



The homology groups of $N(G)$ can be easily calculated using a computer.





$CN(\{1\}) = \{2, 6, 7\}$, $CN(\{2\}) = \{1, 3, 4\}$,
$CN(\{3\}) = \{2, 4\}$, $CN(\{4\}) = \{2, 3, 5\}$, $CN(\{5\}) = \{4, 6\}$,
$CN(\{6\}) = \{1, 5\}$, $CN(\{7\}) = \{1, 8\}$, $CN(\{8\}) = \{7\}$

Proposition

$N(K_n) = \dot{\Delta}_{n-1}$ (the set of all proper faces of a $(n-1)$-simplex Δ_{n-1})

$G = (V, E) \qquad A_1, A_2 \subset V$

$A_1 \uplus A_2 \overset{def.}{=} A_1 \times \{1\} \cup A_2 \times \{2\} \subset V \times \{1, 2\}$

$$B(G) := \{A_1 \uplus A_2 \mid A_1, A_2 \subset V,\ A_1 \cap A_2 = \emptyset,$$
$$\text{"}u \in A_1, v \in A_2 \Rightarrow \{u, v\} \in E\text{"},$$
$$CN(A_1) \neq \emptyset \text{ or } CN(A_2) \neq \emptyset \quad \}$$

Proposition

$|B(G)| \simeq |N(G)|$

$T\colon B(G) \to B(G),\ A_1 \uplus A_2 \mapsto A_2 \uplus A_1 \quad (T^2 = 1)$

$B(G)$ is a free $\mathbb{Z}_2$-space.

Graph homomorphisms

G, H : simple graphs

$f\colon V(G) \to V(H)$ is a <u>graph homomorphism</u>

$\overset{def.}{\Longleftrightarrow}$ "$\{u, v\} \in E(G) \Rightarrow \{f(u), f(v)\} \in E(G)$"

$$\chi(G) = n \Leftrightarrow \begin{cases} \exists \text{ graph hom } G \to K_n \\ \nexists \text{ graph hom } G \to K_{n-1} \end{cases}$$

A graph hom $f\colon V(G) \to V(H)$ induce a simplicial map $B(f)\colon B(G) \to B(H)$, $A_1 \uplus A_2 \mapsto f(A_1) \uplus f(A_2)$.

$B(f)$ is a $\mathbb{Z}_2$-map.

$(B(f)(T(A_1 \uplus A_2)) = B(f)(A_2 \uplus A_1) = f(A_2) \uplus f(A_1) = T(f(A_1) \uplus f(A_2))$
$\quad = T(B(f)(A_1 \uplus A_2)))$

$\chi(G) = n \Rightarrow \exists g\colon |B(G)| \to |B(K_n)|$ $\mathbb{Z}_2$-map

A generalized Borsuk-Ulam theorem

K, L : free $\mathbb{Z}_2$-simplicail complexes (K and L are finite)

K is connected and $H_p(K;\mathbb{Z}/2) = 0$ for $1 \leqq p \leqq n$. $H_p(L;\mathbb{Z}/2) = 0$ for $p \geqq n+1$

Then, there is no $\mathbb{Z}_2$-map from $|K|$ to $|L|$.

$|B(G)| \simeq |N(G)|$, $|B(K_k)| \simeq |N(K_k)| \approx S^{k-2}$

Theorem(Lovász + x)

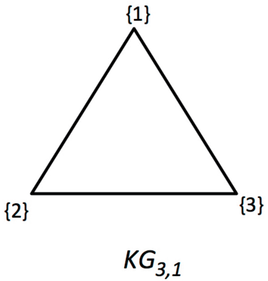
If $N(G)$ is connected and $H_p(N(G);\mathbb{Z}/2) = 0$ for $1 \leqq p \leqq n$, then $\chi(G) \geqq n+3$.

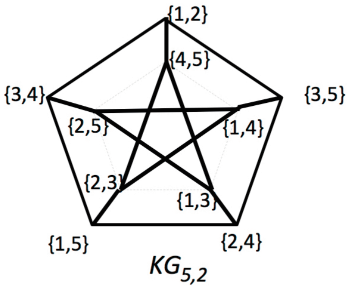
Kneser graph $KG_{n,k}$ $(2k \leqq n)$

$$V(KG_{n,k}) = \binom{[n]}{k} \qquad \left(\binom{[n]}{k} = \{A \subset [n] \mid |A| = k\} \right)$$

$A_1, A_2 \subset V(KG_{n,k})$,

$$\{A_1, A_2\} \in E(KG_{n,k}) \overset{\text{def.}}{\Longleftrightarrow} A_1 \cap A_2 = \emptyset$$





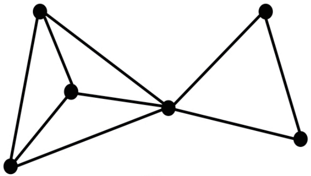
Kneser-Lovász theorem (1978)

$\chi(KG_{n,k}) = n - 2k + 2$

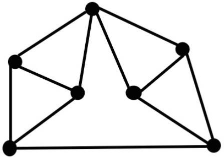
(*proof*) $c\colon V(KG_{n,k}) \to [n-2k+2],\ A \mapsto \min\{\min A, n-2k+2\}$
is a coloring. $(\because \chi(KG_{n,k}) \leqq n-2k+2.)$
Lovász proved that $|N(KG_{n,k})|$ is $(n-2k-1)$-connected.
$\therefore \chi(KG_{n,k}) \geqq (n-2k-1)+3 = n-2k+2.$

Remark. In fact, Lovász consider
$\mathcal{L}(G) = \{(A_0,\dots,A_k) \mid A_i \in N(G), CN^2(A_i) = A_i,\ A_0 \subset \cdots \subset A_k\}$
which is a subcomplex of $sd(N(G))$. $\mathcal{L}(G) \simeq N(G)$. $\mathcal{L}(G)$ has a $\mathbb{Z}_2$-action by CN.

G_1andG_2 are examples such that $H_1(N(G_i);\mathbb{Z}_2) \neq 0$ and $\chi(G_i) = 4 \gneqq 3 + 0$. $(i = 1, 2)$



G_1



G_2

$\mathrm{ind}_{\mathbb{Z}_2} B(G) := \min\{n \mid \exists\ \mathbb{Z}_2\text{-map } f : |B(G)| \to S^n\}$

Proposition. $\chi(G) \geqq \mathrm{ind}_{\mathbb{Z}_2} B(G) + 2$

$\mathrm{ind}_{\mathbb{Z}_2} B(G_1) = 2, \qquad \mathrm{ind}_{\mathbb{Z}_2} B(G_2) = 1,$

G is a triangulation graph of D^2

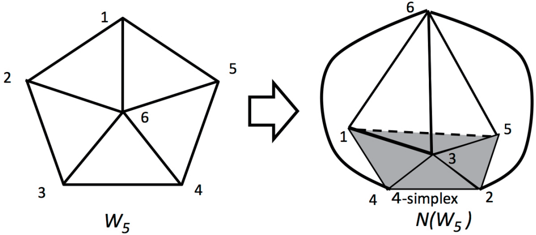
$\overset{\text{def.}}{\Longleftrightarrow}$ $\begin{cases} \exists K : \text{ a triangulation of } D^2(|K| \approx D^2) \\ \quad \text{s.t. } G = K^{(1)} = \{\sigma \in K \mid \dim\sigma \leqq 1\} \end{cases}$

G :triangulation graph of $D^2 \Rightarrow \chi(G) = 3$ or 4.

Theorem(Tomita)

G :triangulation graph of D^2

$|N(G)|$ is k-connected and is not $(k+1)$-connected $\Rightarrow \chi(G) = k + 3$.



Circular chromatic number

$c\colon V \to [n]$ is a <u>(n,k)-coloring</u> $(n \geqq 2k)$

$$\overset{\text{def.}}{\iff} \begin{cases} c \text{ is a coloring} \\ k \leqq |c(x) - c(y)| \leqq n - k \text{ for all } \{x,y\} \in E \end{cases}$$

The <u>circularl chromatic number</u> $\chi_c(G)$ is defined by

$$\chi_c(G) = \inf \left\{ \frac{n}{k} \,\middle|\, \exists c\colon V \to [n]\ (n,k)\text{-coloring} \right\}$$

Proposition(Bondy-Hell)

(1) G has a (p,q)-coloring , $p/q \leqq p'/q'$(p' and q' are positive integers)
$\Rightarrow$ G has a (p',q')-coloring.

(2) $|V(G)| = n$, G has a (p,q)-coloring with $\gcd(p,q) = 1$ and $p > n$
$\Rightarrow$ G has a (p',q')-coloring with $p' < p$ and $p'/q' < p/q$.

Corollary

$$\chi_c(G) = \min\left\{ \frac{p}{q} \mid \exists c\colon V \to [p]\ (p,q)\text{-coloring}, p \leqq |V(G)| \right\}$$

Proposition(Bondy-Hell)

$$\chi(G) - 1 < \chi_c(G) \leqq \chi(G).$$

(*Proof*) $(p,1)$-coloring c is a coloring $c\colon V(G) \to [p]$. $\therefore\ \chi_c(G) \leqq \chi(G)$.
If there exists a (p,q)-coloring such that $p/q \leqq \chi(G) - 1$, then there exists a $(\chi(G)-1, 1)$-coloring by the proposition written in the previous page. $\therefore\ \chi_c(G) > \chi(G) - 1$.

Theorem(P-A, Chen)

$$\chi_c(KG_{n,k}) = \chi(KG_{n,k}).$$

First, Johnson, Holroyd and Stahl studied the circular chromatic number of Kneser graphs, and conjectured that the above equality always holds(1997).

Meunier and Simonyi-Tardos proved that if n is even then $\chi_c(KG_{n,k}) = \chi(KG_{n,k})$, independently(2005, 2006).

Chen completely proved the equality always holds(2011).

Chang, Liu and Zhu give a short proof of this theorem(2013).

We recall

$$\chi(G) = \min\{n \in \mathbb{N} \mid \exists f\colon V \to K_n \text{ graph hom}\}$$

For the circular chromatic number, we define a graph $G_{p,q}$ by.

$V(G_{p,q}) = [p]$

$\{u, v\} \in E(G_{p,q}) \Leftrightarrow q \leqq |u - v| \leqq p - q$

Then, $\chi_c(G) = \min\left\{ \frac{p}{q} \mid \exists f\colon V \to G_{p,q} \text{ graph hom.} \right\}$.

$\exists f\colon G \to G_{p,q}$ graph hom. $\Rightarrow \exists \tilde{f}\colon |B(G)| \to |B(G_{p,q})|$ $\mathbb{Z}_2$-map.

We will study the topology of $|B(G_{p,q})| (\simeq |N(G_{p,q})|)$.

Theorem

$p > 2q$

In case $2q \nmid p$,

$$H_k(N(G_{p,q});\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & (k = 0 \text{ or } k = 2\lfloor \frac{p}{2q} \rfloor - 1)) \\ 0 & (k \neq 0, \lfloor \frac{p}{2q} \rfloor - 1) \end{cases}$$

In case $2q \mid p$,

$$H_k(N(G_{p,q});\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & (k = 0) \\ \mathbb{Z}^{2q-1} & (k = \frac{p}{q} - 2) \\ 0 & (k \neq 0, \frac{p}{q} - 2) \end{cases}$$

Proposition

$\chi(G)$ is even($\chi(G) \geqq 4$), $N(G)$ is connected and $H_p(N(G); \mathbb{Z}/2) = 0$ for $1 \leqq p \leqq \chi(G) - 3$,

$\Longrightarrow \chi_c(G) = \chi(G)$.

(*Proof*) Put $n = \chi(G)$ and $\frac{p}{q} = \chi_c(G)$. Assume that $\frac{p}{q} < n$.

There is a graph homomorphism $f\colon V(G) \to V(G_{p.q})$.

We have the induced $\mathbb{Z}_2$-map $B(f)\colon B(G) \to B(G_{p.q})$.

Because $B(G)$ is connected and $H_p(B(G); \mathbb{Z}/2) \cong H_p(N(G); \mathbb{Z}/2) = 0$ for $1 \leqq p \leqq n - 3$, there exist a integer k such that $k \geqq n - 2$ and $H_k(N(G_{p,q}); \mathbb{Z}/2) \cong H_k(B(G_{p,q}); \mathbb{Z}/2) \neq 0$ (by a generalized Borsuk-Ulam theorem).

Because n is even and $\frac{p}{q} < n$, $\lfloor \frac{p}{2q} \rfloor = \frac{n}{2} - 1$. Therefore $H_k(N(G_{p,q}); \mathbb{Z}/2) = 0$ for $k \geqq n - 2$. This is contradiction. □

For Kneser graph $KG_{n,k}$ $(n > 2k)$, $\chi(KG_{n,k}) = n - 2k + 2$ and $H_p(N(KG_{n,k}); \mathbb{Z}/2) = 0$ for $1 \leqq p \leqq n - 2k - 1$.
Therefore if n is even, $\chi_c(KG_{n,k}) = \chi(KG_{n,k})$.

There exists a graph G such that $N(G)$ is connected, $H_p(N(G); \mathbb{Z}/2) = 0$ for $1 \leqq p \leqq \chi(G) - 3$ and $\chi_c(G) < \chi(G)$. ($\chi(G)$ is odd.)
$\chi(G_{9,2}) = 5$, $H_k(N(G_{9,2}); \mathbb{Z}/2) = 0$ for $k = 1, 2$,
$H_3(N(G_{9,2}); \mathbb{Z}/2) \cong \mathbb{Z}/2\mathbb{Z}$
$\chi_c(G_{9,2}) = \frac{9}{2} < \chi(G_{9,2})$.