Homology groups of neighborhood complexes of graphs
(Introduction to topological combinatorics)

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Chromatic number

\( G = (V, E) \) : simple graph (no loop, no multi edge)

\([n] := \{1, 2, \ldots, n\}\)

\(c : V \to [n] \) is a coloring \( \overset{\text{def.}}{\iff} \{u, v\} \in E \Rightarrow c(u) \neq c(v)\)

We define the chromatic number \( \chi(G) \) of \( G \) by

\[ \chi(G) = \min \{ n \in \mathbb{N} \mid \exists c : V \to [n] \text{ coloring} \} \]

We define the complete graph \( K_n \) by

\[ V(K_n) = [n], \ E(K_n) = \{\{u, v\} \mid u, v \in [n], u \neq v\} \ . \]

We can easily see \( \chi(K_n) = n \).

Let \( C_n \) be a graph such that

\[ V(C_n) = [n], \ E(C_n) = \{\{1, 2\}, \{2, 3\}, \ldots, \{n - 1, n\}, \{n, 1\}\} \ . \]

\( n \) is even \( \Rightarrow \) \( \chi(C_n) = 2 \). \( n \) is odd \( \Rightarrow \) \( \chi(C_n) = 3 \).
Neighborhood complexes

\[ G = (V, E) : \text{simple graph (no loop, no multi edge)} \]
\[ A \subseteq V \]
\[ CN(A) := \{v \mid \{v, a\} \in E \text{ for } \forall a \in A\} \]
\[ N(G) := \{A \subseteq V \mid CN(A) \neq \emptyset\} \text{ (neighborhood complex)} \]
\[ N(G) \text{ is a simplicial complex.} \]
\[ \{A \mid A \text{ is a maximal face of } N(G)\} \subset \{CN(\{v\}) \mid v \in V(G)\} \]

The homology groups of \( N(G) \) can be easily calculated using a computer.
\[ CN(\{1\}) = \{2, 6, 7\}, \quad CN(\{2\}) = \{1, 3, 4\}, \]
\[ CN(\{3\}) = \{2, 4\}, \quad CN(\{4\}) = \{2, 3, 5\}, \quad CN(\{5\}) = \{4, 6\}, \]
\[ CN(\{6\}) = \{1, 5\}, \quad CN(\{7\}) = \{1, 8\}, \quad CN(\{8\}) = \{7\} \]

**Proposition**

\[ N(K_n) = \Delta_{n-1} \text{ (the set of all proper faces of a } (n - 1)\text{-simplex } \Delta_{n-1}) \]
Box complexes

\[
G = (V, E) \quad A_1, A_2 \subset V
\]

\[
A_1 \uplus A_2 \overset{\text{def.}}{=} A_1 \times \{1\} \cup A_2 \times \{2\} \subset V \times \{1, 2\}
\]

\[
B(G) := \{A_1 \uplus A_2 \mid A_1, A_2 \subset V, A_1 \cap A_2 = \emptyset, \quad "u \in A_1, v \in A_2 \Rightarrow \{u, v\} \in E", \quad CN(A_1) \neq \emptyset \text{ or } CN(A_2) \neq \emptyset \}
\]

**Proposition**

\[
|B(G)| \sim |N(G)|
\]

\[
T : B(G) \to B(G), \quad A_1 \uplus A_2 \mapsto A_2 \uplus A_1 \quad (T^2 = 1)
\]

\[
B(G) \text{ is a free } \mathbb{Z}_2\text{-space.}
\]
Graph homomorphisms

$G, H : \text{simple graphs}$

$f : V(G) \to V(H)$ is a graph homomorphism

\[ \text{def} \quad \{u, v\} \in E(G) \Rightarrow \{f(u), f(v)\} \in E(G) \]

\[ \chi(G) = n \iff \exists \text{ graph hom } G \to K_n \]

\[ \not\exists \text{ graph hom } G \to K_{n-1} \]

A graph hom $f : V(G) \to V(H)$ induce a simplicial map

$B(f) : B(G) \to B(H), A_1 \uplus A_2 \mapsto f(A_1) \uplus f(A_2)$.

$B(f)$ is a $\mathbb{Z}_2$-map.

\[ (B(f)(T(A_1 \uplus A_2))) = B(f)(A_2 \uplus A_1) = f(A_2) \uplus f(A_1) = T(f(A_1) \uplus f(A_2)) = T(B(f)(A_1 \uplus A_2)) \]

\[ \chi(G) = n \Rightarrow \exists g : |B(G)| \to |B(K_n)| \text{ $\mathbb{Z}_2$-map} \]
A generalized Borsuk-Ulam theorem

\( K, L : \) free \( \mathbb{Z}_2 \)-simplicial complexes (\( K \) and \( L \) are finite)

\( K \) is connected and \( H_p(K; \mathbb{Z}/2) = 0 \) for \( 1 \leq p \leq n \). \( H_p(L; \mathbb{Z}/2) = 0 \) for \( p \geq n + 1 \)

Then, there is no \( \mathbb{Z}_2 \)-map from \( |K| \) to \( |L| \).

\(|B(G)| \approx |N(G)|, \ |B(K_k)| \approx |N(K_k)| \approx S^{k-2} \)

Theorem (Lovász + x)

If \( N(G) \) is connected and \( H_p(N(G); \mathbb{Z}/2) = 0 \) for \( 1 \leq p \leq n \), then \( \chi(G) \geq n + 3 \).
Kneser-Lovász theorem

**Kneser graph** $KG_{n,k}$ \hspace{1em} (2k $\leq$ n)

$V(KG_{n,k}) = \binom{[n]}{k}$ \hspace{1em} $\binom{[n]}{k} = \{A \subset [n] \mid |A| = k\}$

$A_1, A_2 \subset V(KG_{n,k})$, \hspace{1em} \{A_1, A_2\} $\in E(KG_{n,k}) \quad \overset{\text{def.}}{\iff} \quad A_1 \cap A_2 = \emptyset$

\[ KG_{3,1} \hspace{4em} KG_{5,2} \]
Kneser-Lovász theorem (1978)

\[ \chi(KG_{n,k}) = n - 2k + 2 \]

(proof) c: \( V(KG_{n,k}) \rightarrow [n - 2k + 2] \), \( A \mapsto \min\{\min A, n - 2k + 2\} \)
is a coloring. (\( \therefore \chi(KG_{n,k}) \leq n - 2k + 2 \).)

Lovász proved that \( |N(KG_{n,k})| \) is \( (n - 2k - 1) \)-connected.
\( \therefore \chi(KG_{n,k}) \geq (n - 2k - 1) + 3 = n - 2k + 2 \).

Remark. In fact, Lovász consider
\[ \mathcal{L}(G) = \{(A_0, \ldots, A_k) \mid A_i \in N(G), CN^2(A_i) = A_i, A_0 \subset \cdots \subset A_k\} \]
which is a subcomplex of \( sd(N(G)) \). \( \mathcal{L}(G) \simeq N(G) \). \( \mathcal{L}(G) \) has a \( \mathbb{Z}_2 \)-action
by \( CN \).
$G_1$ and $G_2$ are examples such that $H_1(N(G_i);\mathbb{Z}_2) \neq 0$ and
\[\chi(G_i) = 4 \geq 3 + 0. \; (i = 1, 2)\]

\[
\begin{align*}
\text{ind}_{\mathbb{Z}_2} B(G) &:= \min\{n \mid \exists \; \mathbb{Z}_2\text{-map } f : |B(G)| \to S^n\} \\
\text{Proposition.} \quad \chi(G) &\geq \text{ind}_{\mathbb{Z}_2} B(G) + 2 \\
\text{ind}_{\mathbb{Z}_2} B(G_1) &= 2, \quad \text{ind}_{\mathbb{Z}_2} B(G_2) = 1,
\end{align*}
\]
G is a triangulation graph of $D^2$

\[ \exists K : \text{a triangulation of } D^2 (|K| \approx D^2) \]

s.t. $G = K^{(1)} = \{ \sigma \in K | \dim \sigma \leq 1 \}$

$G$ : triangulation graph of $D^2 \Rightarrow \chi(G) = 3$ or $4$.

**Theorem (Tomita)**

$G$ : triangulation graph of $D^2$

$|N(G)|$ is $k$-connected and is not $(k + 1)$-connected \( \Rightarrow \chi(G) = k + 3 \).
Circular chromatic number

c: \( V \rightarrow [n] \) is a \((n, k)\)-coloring \((n \geq 2k)\)

\[
\begin{align*}
def. \quad c \text{ is a coloring} \\
\text{such that} \\
k \leq |c(x) - c(y)| \leq n - k \text{ for all } \{x, y\} \in E
\end{align*}
\]

The circular chromatic number \( \chi_c(G) \) is defined by

\[
\chi_c(G) = \inf \left\{ \frac{n}{k} \mid \exists c: V \rightarrow [n] \text{ } (n, k)\text{-coloring} \right\}
\]

**Proposition (Bondy-Hell)**

1. If \( G \) has a \((p, q)\)-coloring, \( p/q \leq p'/q' \) (\( p' \) and \( q' \) are positive integers)

\[ \Rightarrow \] \( G \) has a \((p', q')\)-coloring.

2. If \( |V(G)| = n \), \( G \) has a \((p, q)\)-coloring with \( \gcd(p, q) = 1 \) and \( p > n \)

\[ \Rightarrow \] \( G \) has a \((p', q')\)-coloring with \( p' < p \) and \( p'/q' < p/q \).
Corollary

\[ \chi_c(G) = \min \left\{ \frac{p}{q} \mid \exists c : V \rightarrow [p] \text{ (p, q)-coloring, } p \leq |V(G)| \right\} \]

Proposition (Bondy-Hell)

\[ \chi(G) - 1 < \chi_c(G) \leq \chi(G). \]

(Proof) \((p, 1)\)-coloring \(c\) is a coloring \(c : V(G) \rightarrow [p]\). \(\therefore \chi_c(G) \leq \chi(G)\). If there exists a \((p, q)\)-coloring such that \(p/q \leq \chi(G) - 1\), then there exists a \((\chi(G) - 1, 1)\)-coloring by the proposition written in the previous page. \(\therefore \chi_c(G) > \chi(G) - 1\).
Theorem (P-A, Chen)

\[ \chi_c(KG_{n,k}) = \chi(KG_{n,k}). \]

First, Johnson, Holroyd and Stahl studied the circular chromatic number of Kneser graphs, and conjectured that the above equality always holds (1997). Meunier and Simonyi-Tardos proved that if \( n \) is even then 
\[ \chi_c(KG_{n,k}) = \chi(KG_{n,k}), \] independently (2005, 2006). Chen completely proved the equality always holds (2011). Chang, Liu and Zhu give a short proof of this theorem (2013).
We recall

\[ \chi(G) = \min \{ n \in \mathbb{N} | \exists f : V \to K_n \text{ graph hom} \} \]

For the circular chromatic number, we define a graph \( G_{p,q} \) by.

\[ V(G_{p,q}) = [p] \]

\[ \{u, v\} \in E(G_{p,q}) \iff q \leq |u - v| \leq p - q \]

Then, \( \chi_c(G) = \min \left\{ \frac{p}{q} | \exists f : V \to G_{p,q} \text{ graph hom.} \right\} \).

\[ \exists f : G \to G_{p,q} \text{ graph hom.} \Rightarrow \exists \tilde{f} : |B(G)| \to |B(G_{p,q})| \mathbb{Z}_2\text{-map.} \]

We will study the topology of \( |B(G_{p,q})| (\sim |N(G_{p,q})|) \).
Theorem

\[ p > 2q \]

In case \( 2q \nmid p \),

\[
H_k(N(G_p,q); \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & (k = 0 \text{ or } k = 2\lfloor \frac{p}{2q} \rfloor - 1) \\ 0 & (k \neq 0, \lfloor \frac{p}{2q} \rfloor - 1) \end{cases}
\]

In case \( 2q \mid p \),

\[
H_k(N(G_p,q); \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & (k = 0) \\ \mathbb{Z}^{2q-1} & (k = \frac{p}{q} - 2) \\ 0 & (k \neq 0, \frac{p}{q} - 2) \end{cases}
\]
Proposition

\( \chi(G) \) is even (\( \chi(G) \geq 4 \)), \( N(G) \) is connected and \( H_p(N(G); \mathbb{Z}/2) = 0 \) for \( 1 \leq p \leq \chi(G) - 3 \),
\[ \implies \chi_c(G) = \chi(G). \]

(Proof) Put \( n = \chi(G) \) and \( \frac{p}{q} = \chi_c(G) \). Assume that \( \frac{p}{q} < n \).

There is a graph homomorphism \( f: V(G) \to V(G_{p,q}) \).

We have the induced \( \mathbb{Z}_2 \)-map \( B(f): B(G) \to B(G_{p,q}) \).

Because \( B(G) \) is connected and \( H_p(B(G); \mathbb{Z}/2) \cong H_p(N(G); \mathbb{Z}/2) = 0 \) for \( 1 \leq p \leq n - 3 \), there exist a integer \( k \) such that \( k \geq n - 2 \) and \( H_k(N(G_{p,q}); \mathbb{Z}/2) \cong H_k(B(G_{p,q}); \mathbb{Z}/2) \neq 0 \) (by a generalized Borsuk-Ulam theorem).

Because \( n \) is even and \( \frac{p}{q} < n \), \( \left\lfloor \frac{p}{2q} \right\rfloor = \frac{n}{2} - 1 \). Therefore \( H_k(N(G_{p,q}); \mathbb{Z}/2) = 0 \) for \( k \geq n - 2 \). This is contradiction. \( \square \)
For Kneser graph $KG_{n,k}$ ($n > 2k$), $\chi(KG_{n,k}) = n - 2k + 2$ and $H_p(N(KG_{n,k}); \mathbb{Z}/2) = 0$ for $1 \leq p \leq n - 2k - 1$.

Therefore if $n$ is even, $\chi_c(KG_{n,k}) = \chi(KG_{n,k})$.

There exists a graph $G$ such that $N(G)$ is connected, $H_p(N(G); \mathbb{Z}/2) = 0$ for $1 \leq p \leq \chi(G) - 3$ and $\chi_c(G) < \chi(G)$. ($\chi(G)$ is odd.)

$\chi(G_9,2) = 5$, $H_k(N(G_9,2); \mathbb{Z}/2) = 0$ for $k = 1, 2$,

$H_3(N(G_9,2); \mathbb{Z}/2) \cong \mathbb{Z}/2\mathbb{Z}$

$\chi_c(G_9,2) = \frac{9}{2} < \chi(G_9,2)$. 