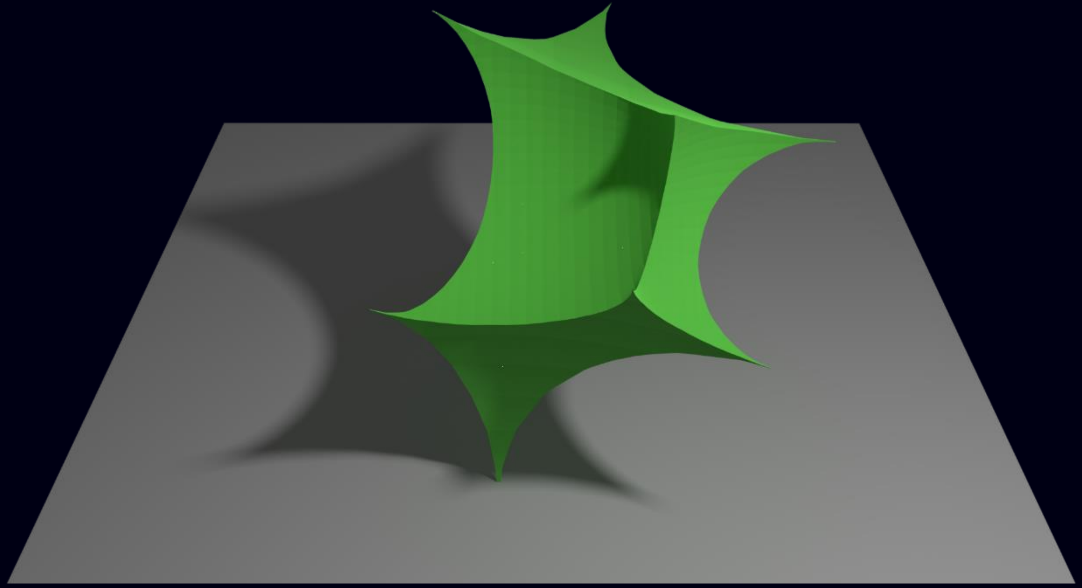
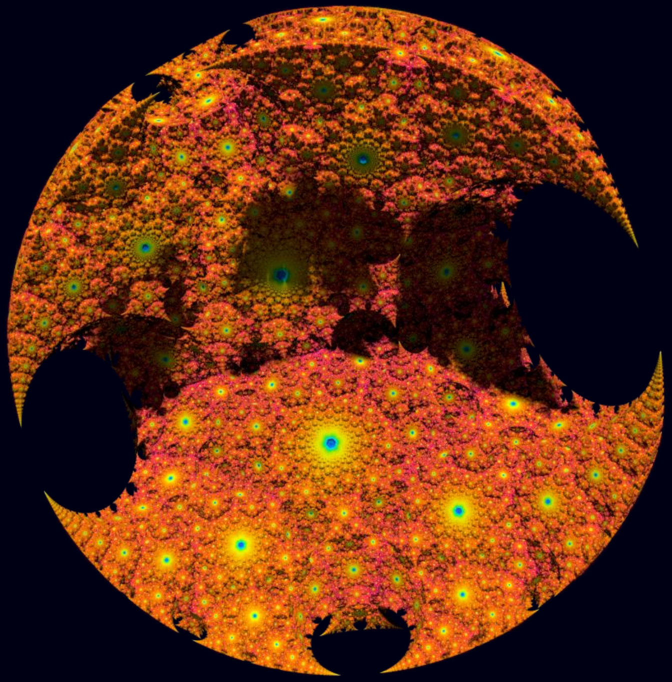
Topology and Computer 2017





Polyhedra with Spherical Faces and Quasi-Fuchsian Fractals

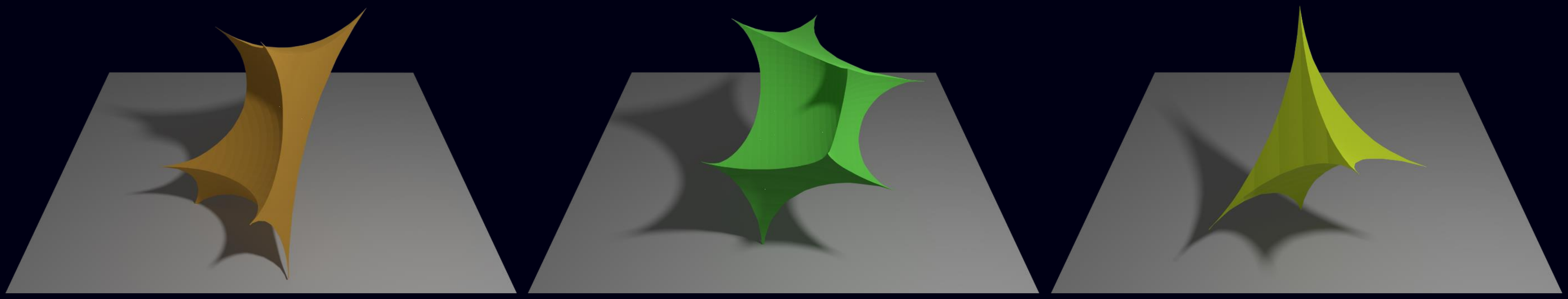
Kento Nakamura

Graduate School of Advanced Mathematical Science, Meiji University

Sphairahedron

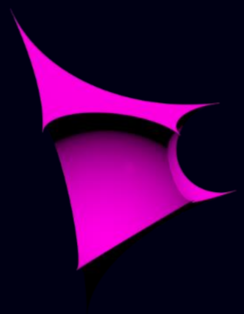
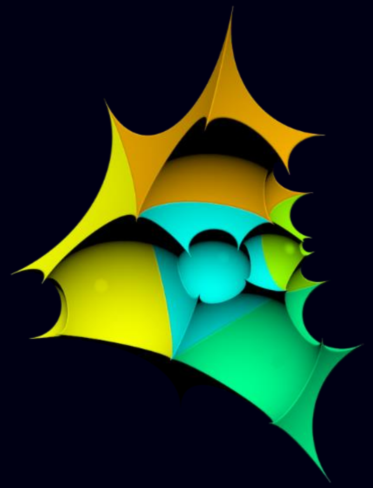
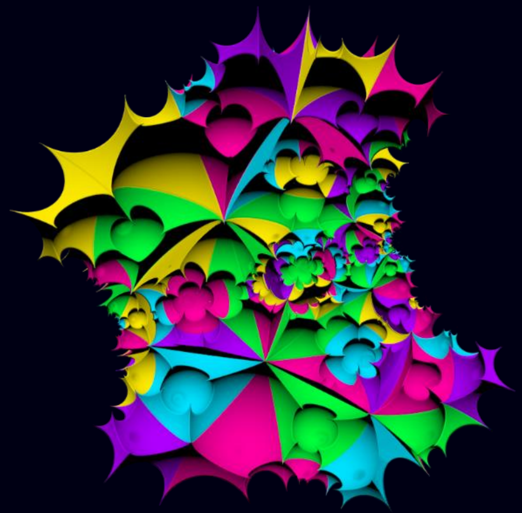
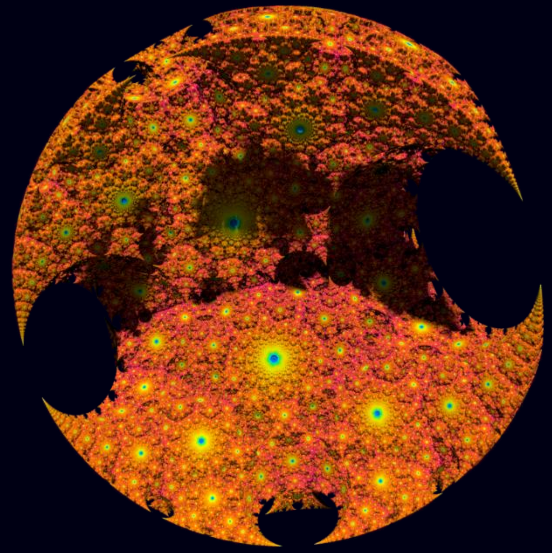
'sphaira-' (= spherical) + '-hedron' (= polyhedron)

New geometrical concept invented by Kazushi Ahara and Yoshiaki Araki (2003)



Quasi-sphere

One of the early examples of the 3-dimensional fractals

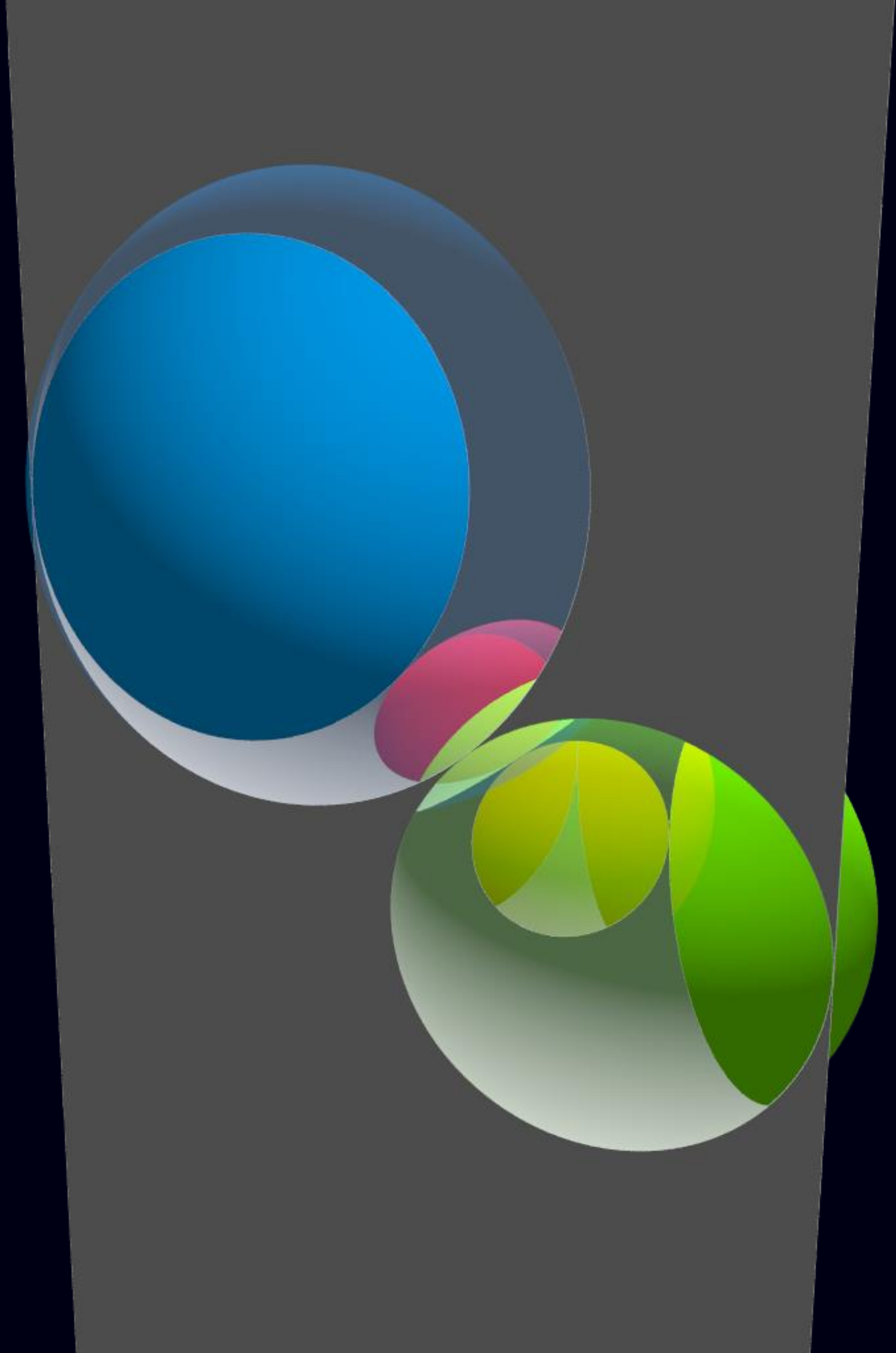
Sphairahedron

Sphairahedron

$S^3 = R^3 \cup \{\infty\}$

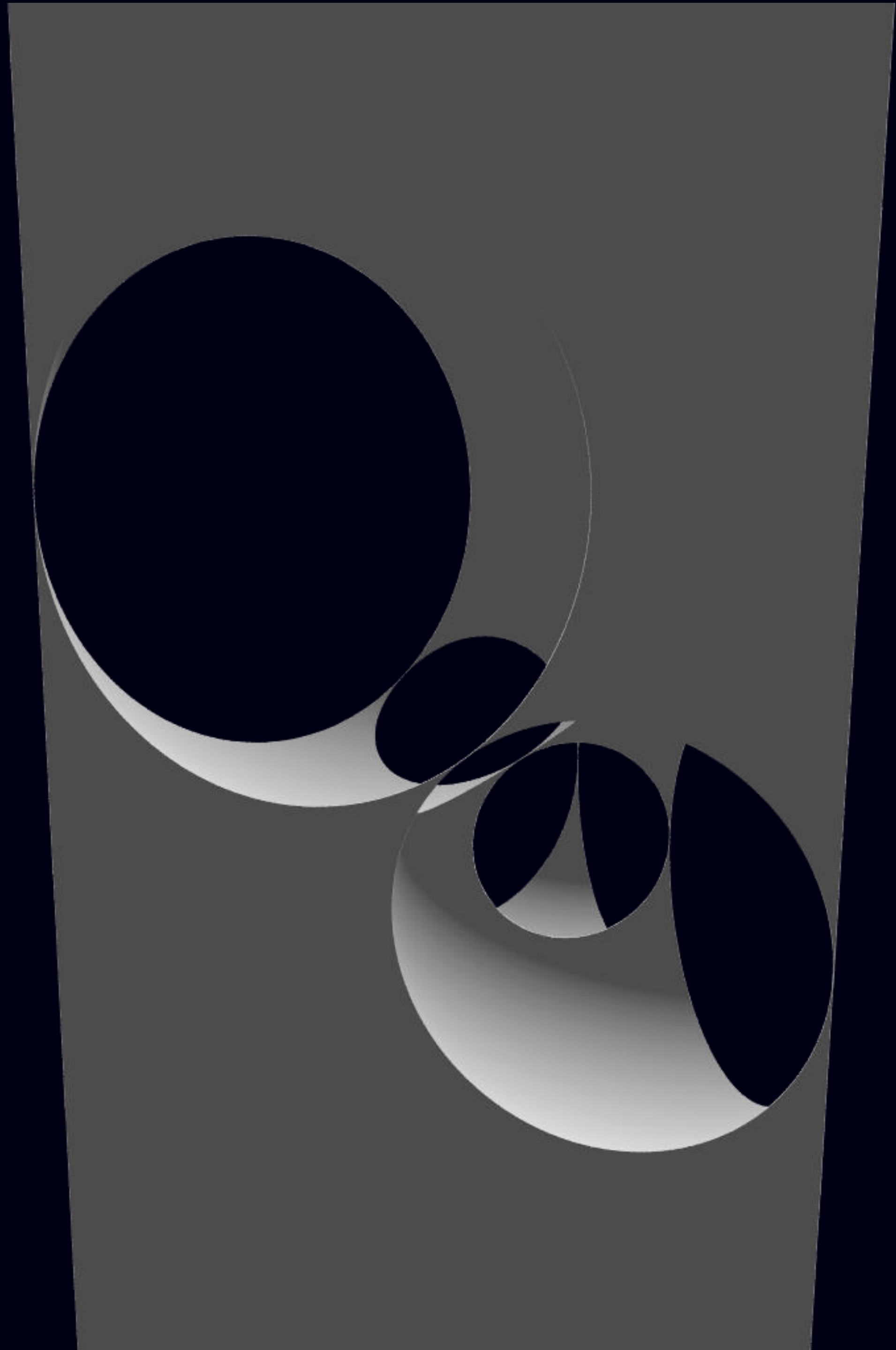
closed-ball: $O_1, O_2, \ldots, O_n$

$A = S^3 - (O_1 \cup O_2 \ldots \cup O_n)$



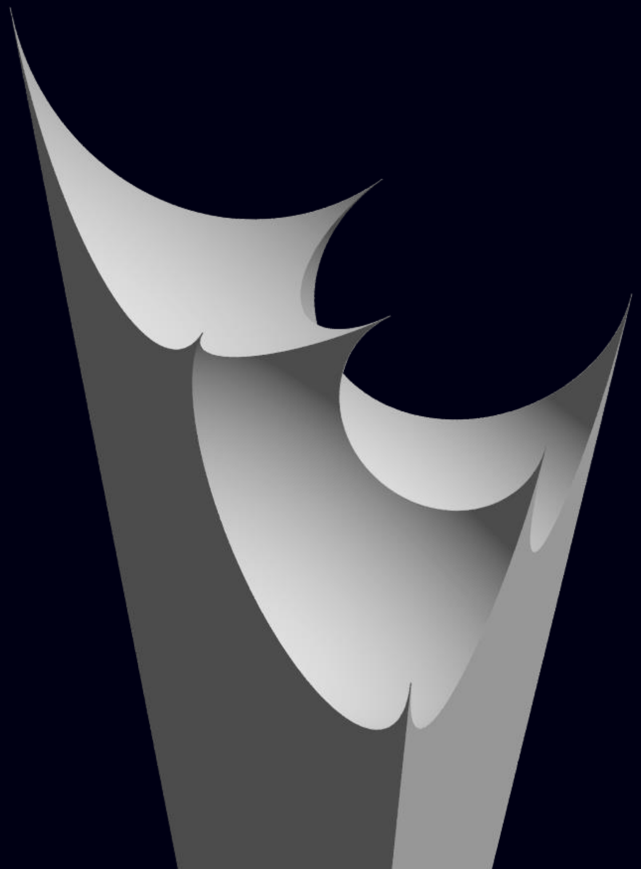
Sphairahedron

One side of the simply connected two components of A



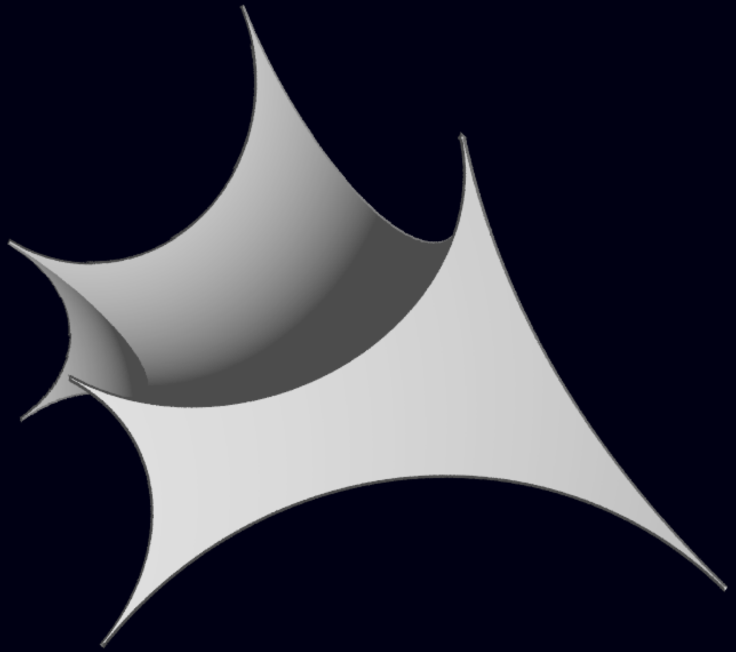
Sphairahedron

One side of the simply connected two components of A



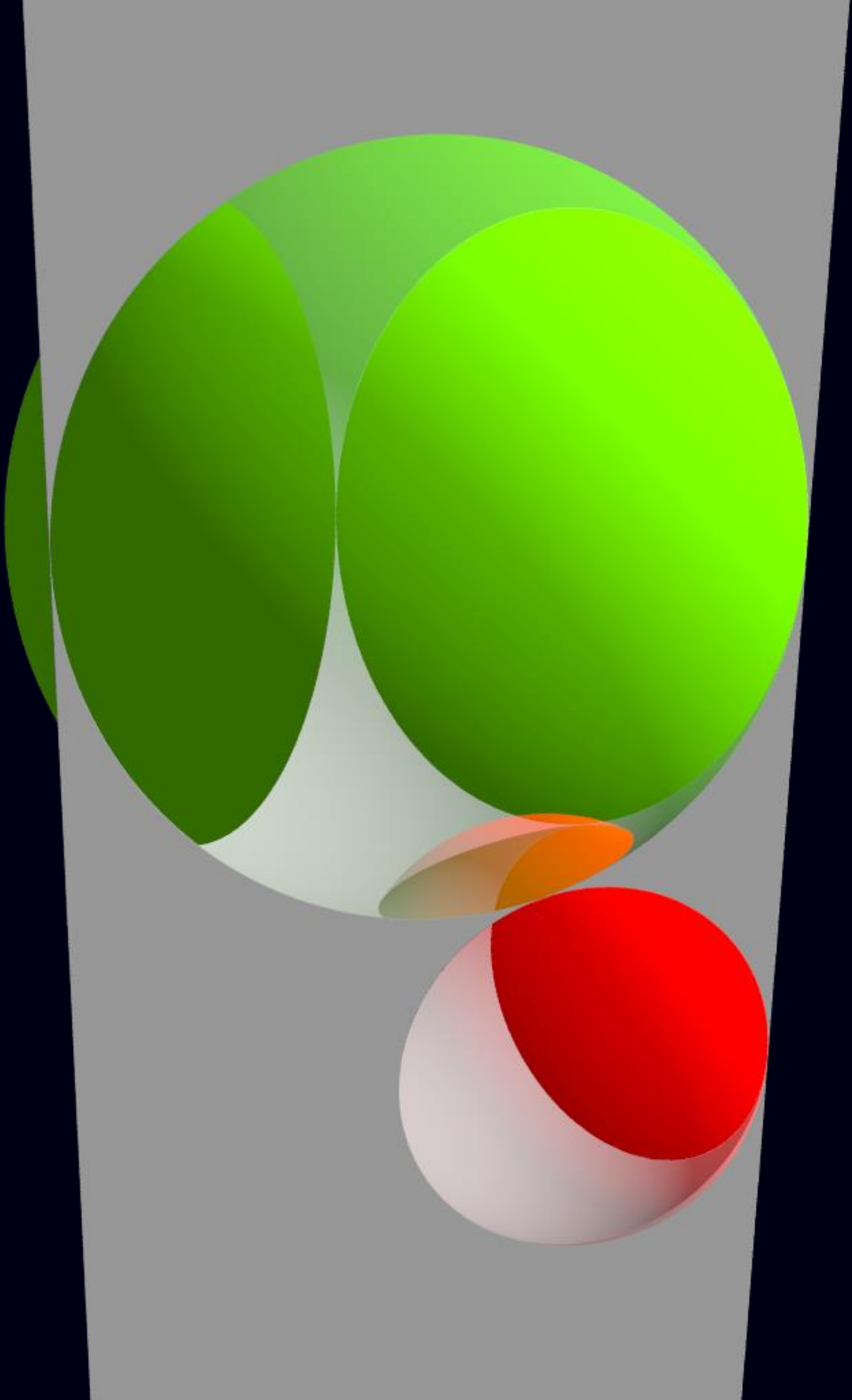
Sphairahedron

One side of the simply connected two components of A



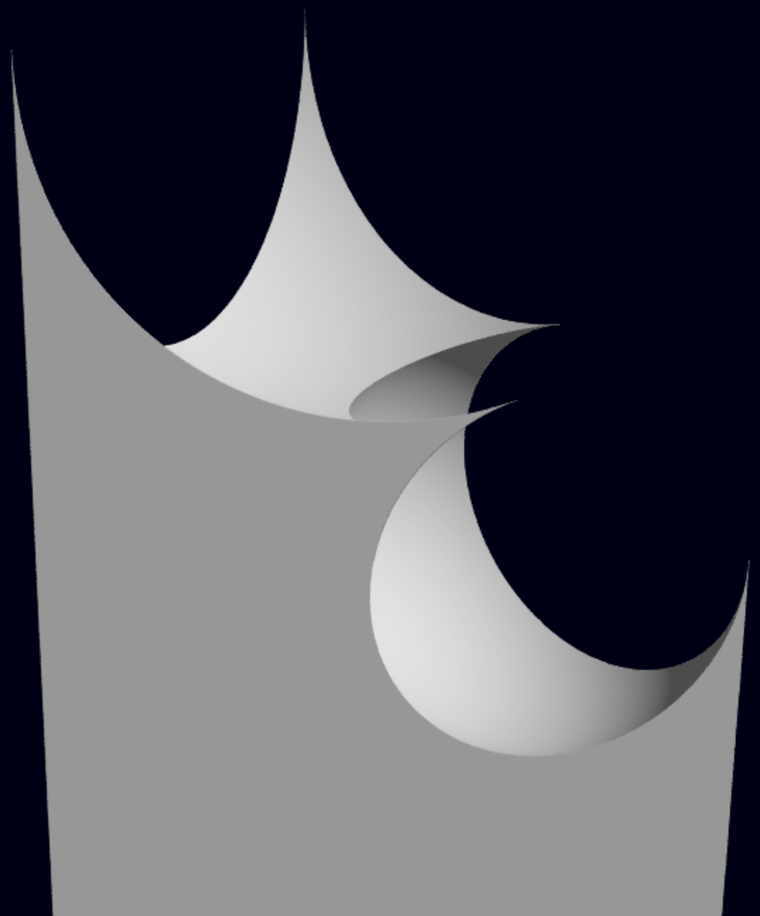
Semi-Sphairahedron

One side of the simply connected three or more components of A



Semi-Sphairahedron

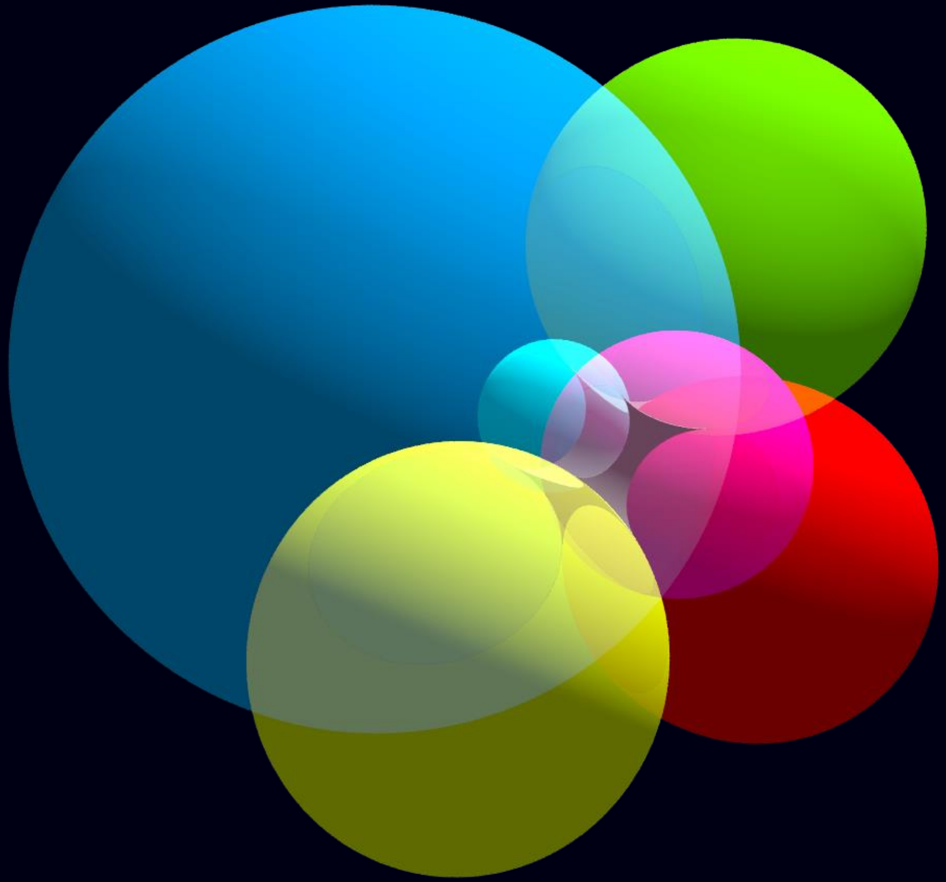
One side of the simply connected three or more components of A



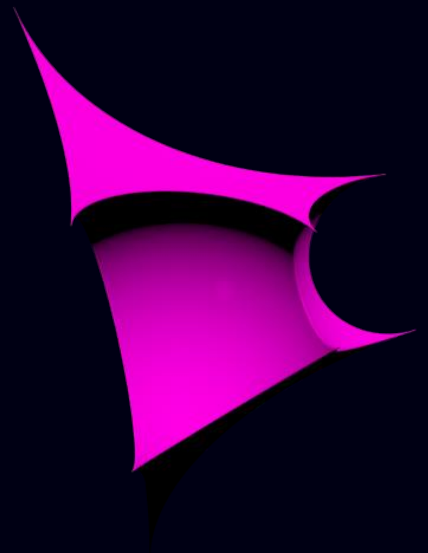
Sphairahedron Group

f_i: *Inversion in* O_i

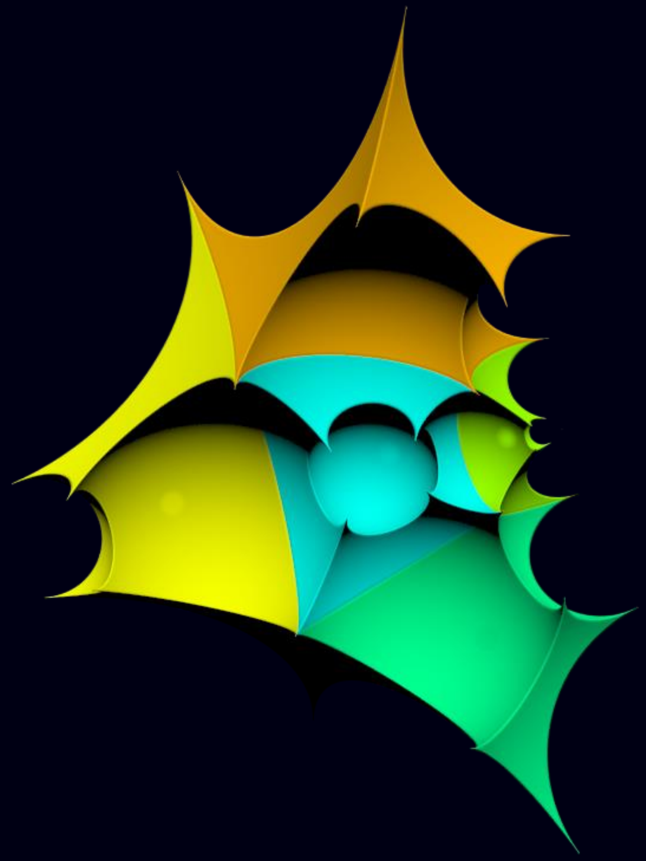
$G = < f_0, f_1, \dots, f_n >$



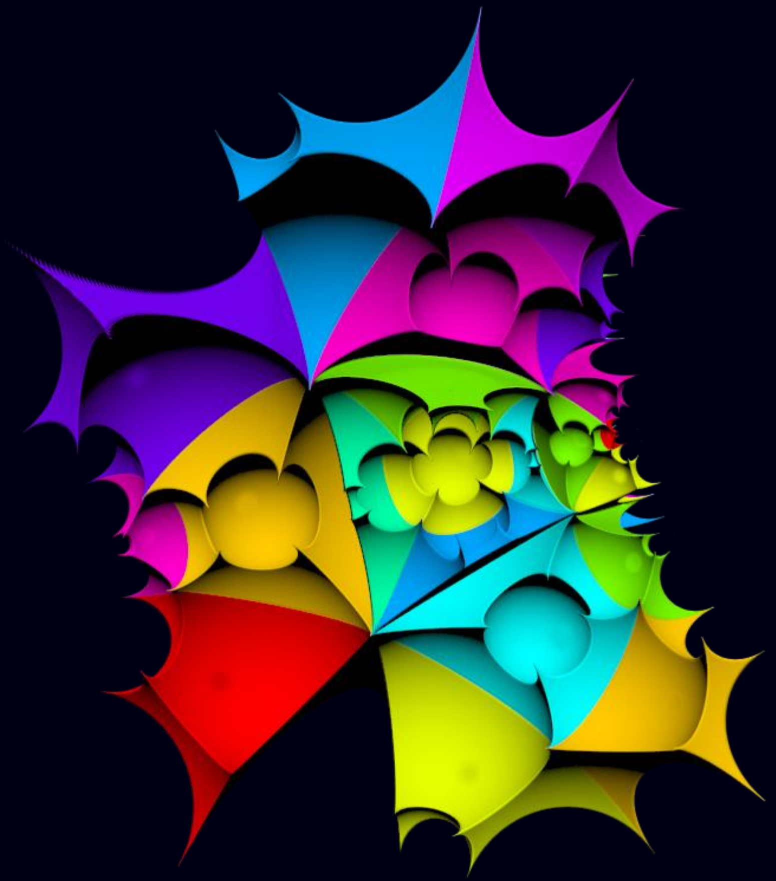
Tessellation by G



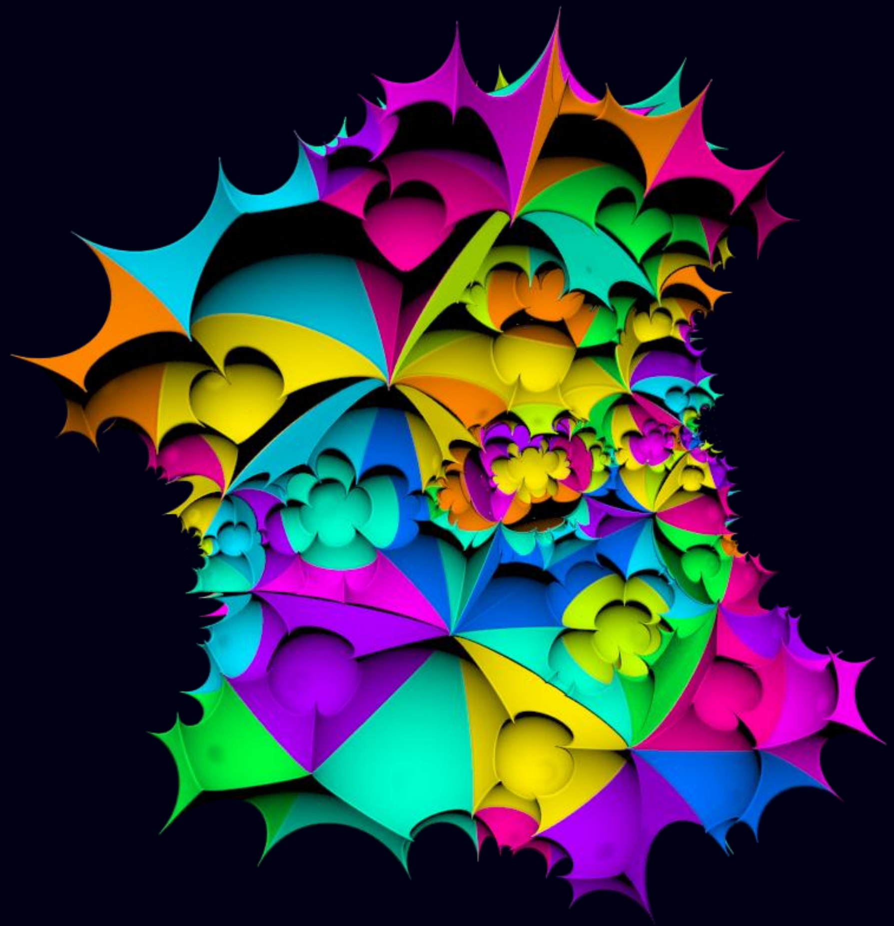
Tessellation by G



Tessellation by G



Tessellation by G



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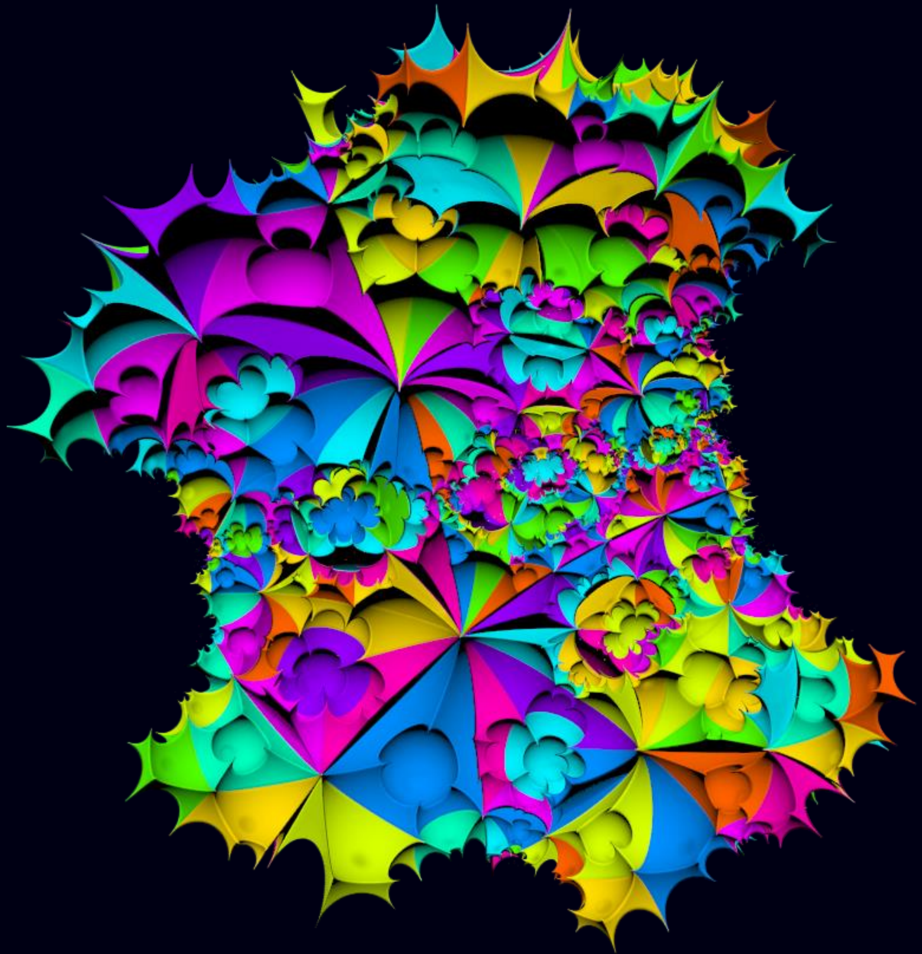
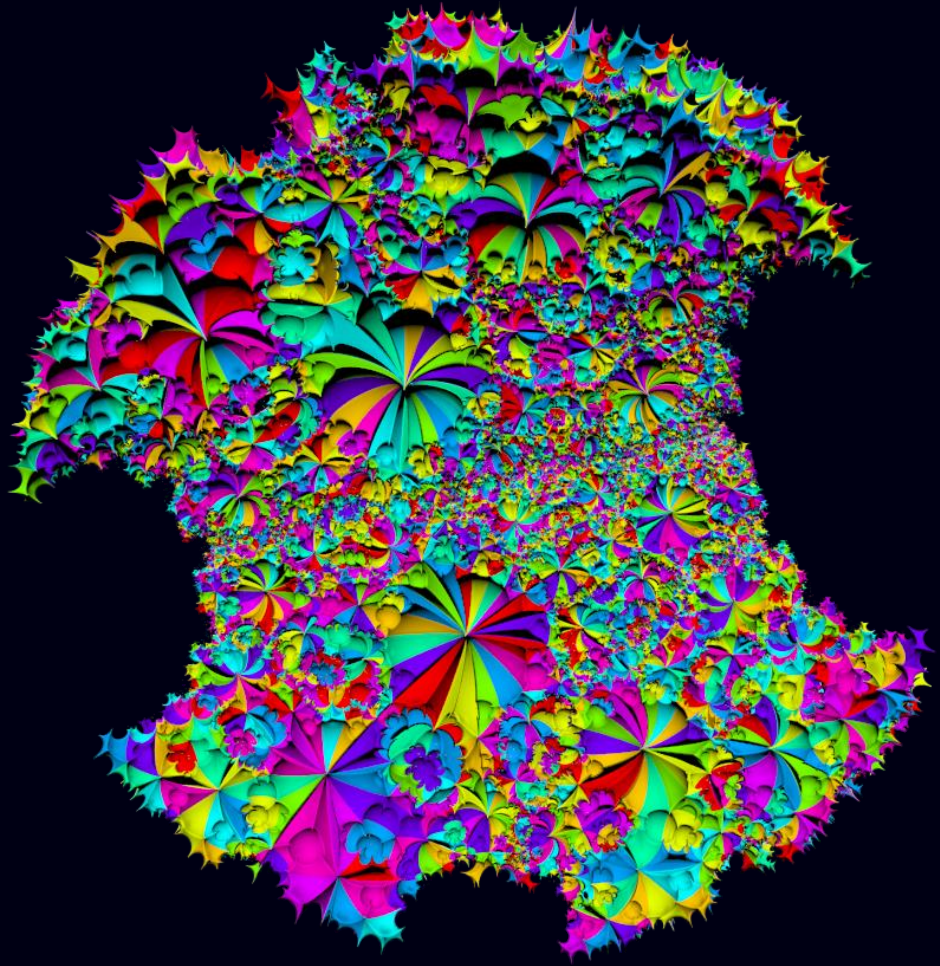


Tessellation by G



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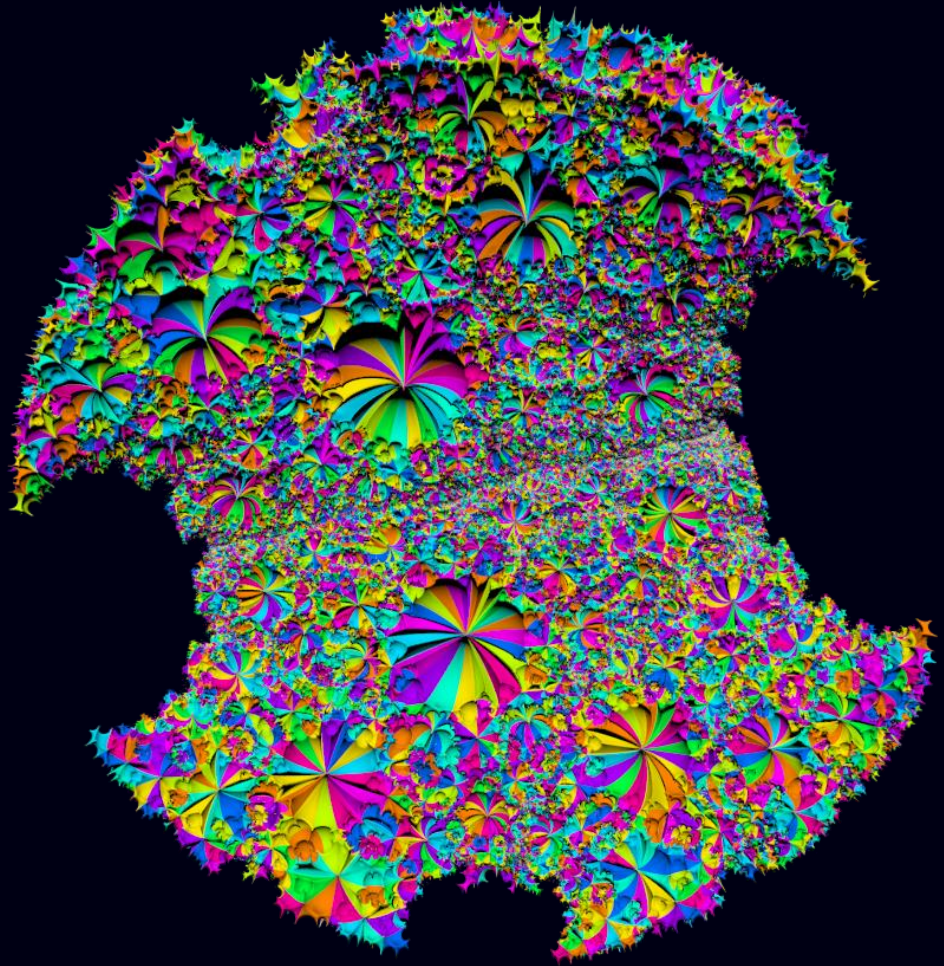

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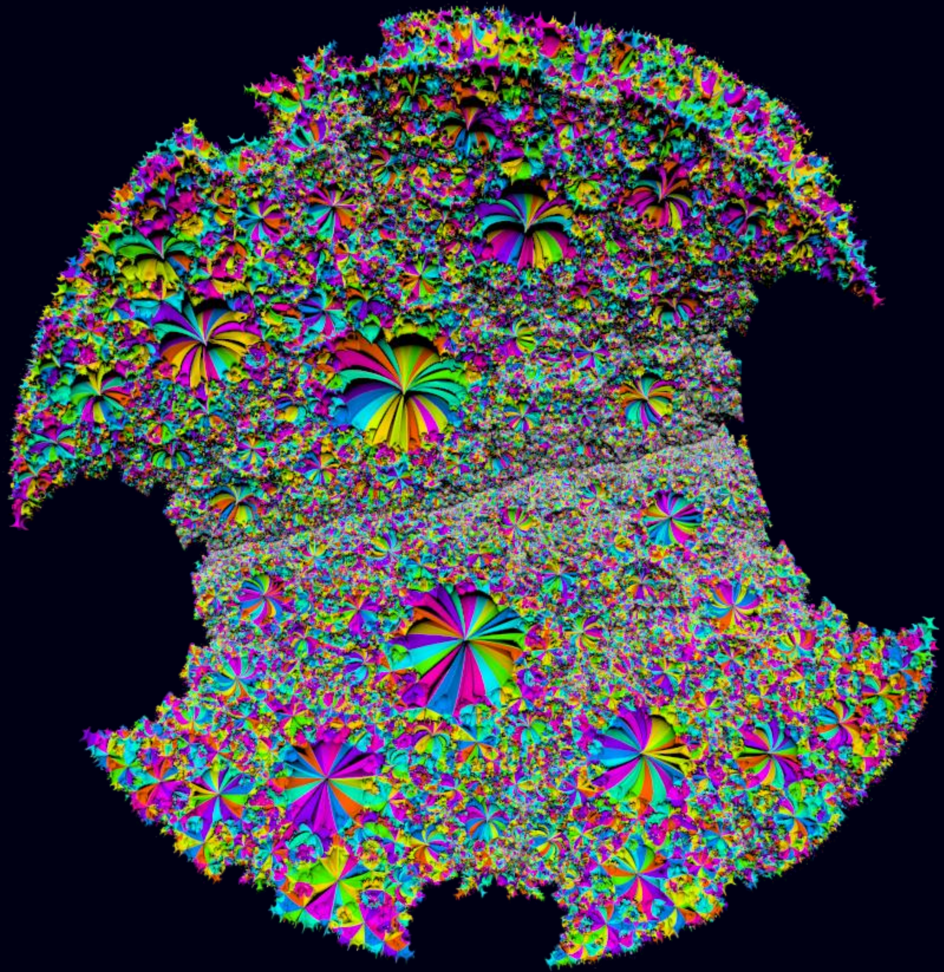

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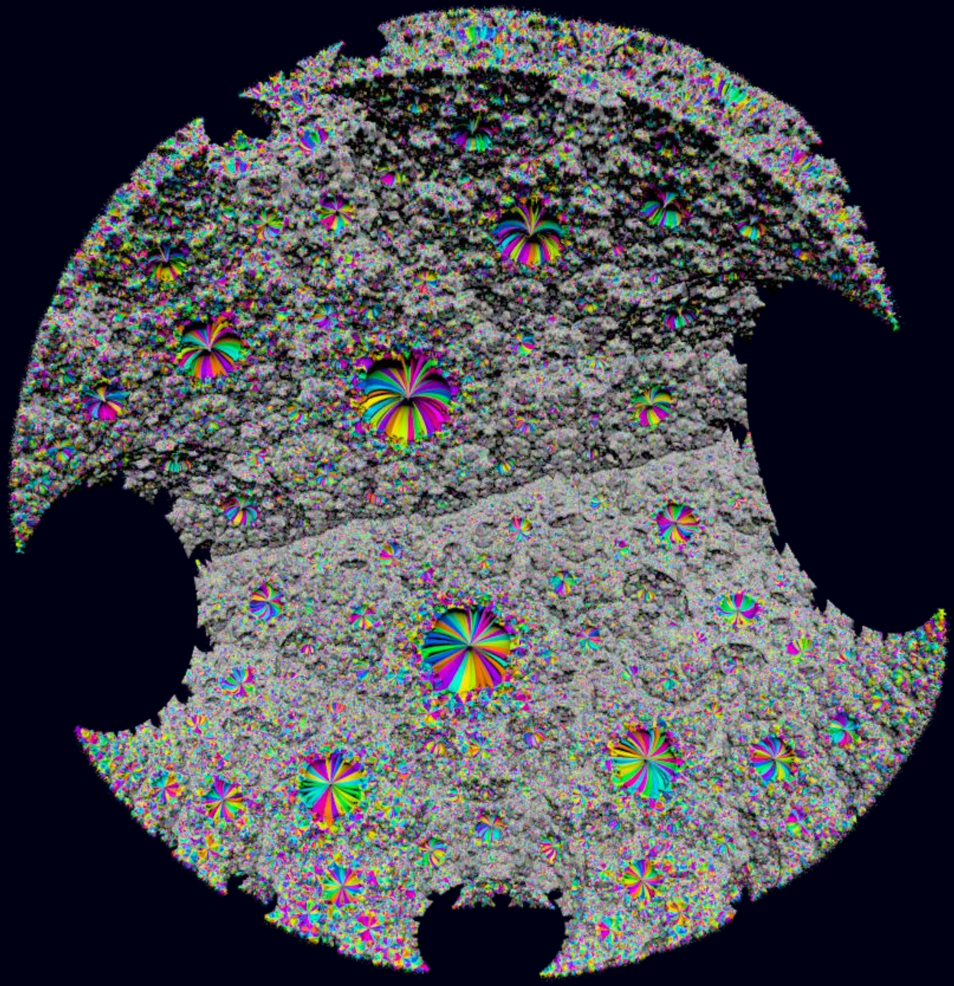
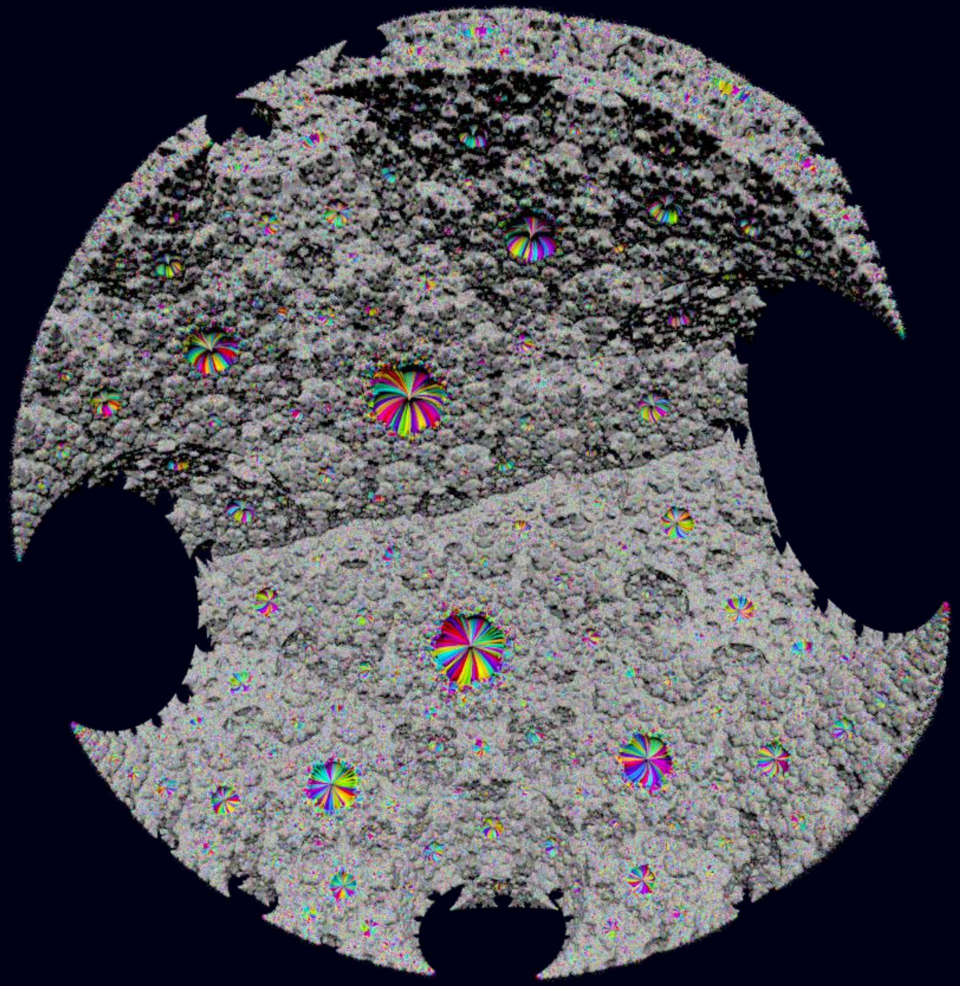
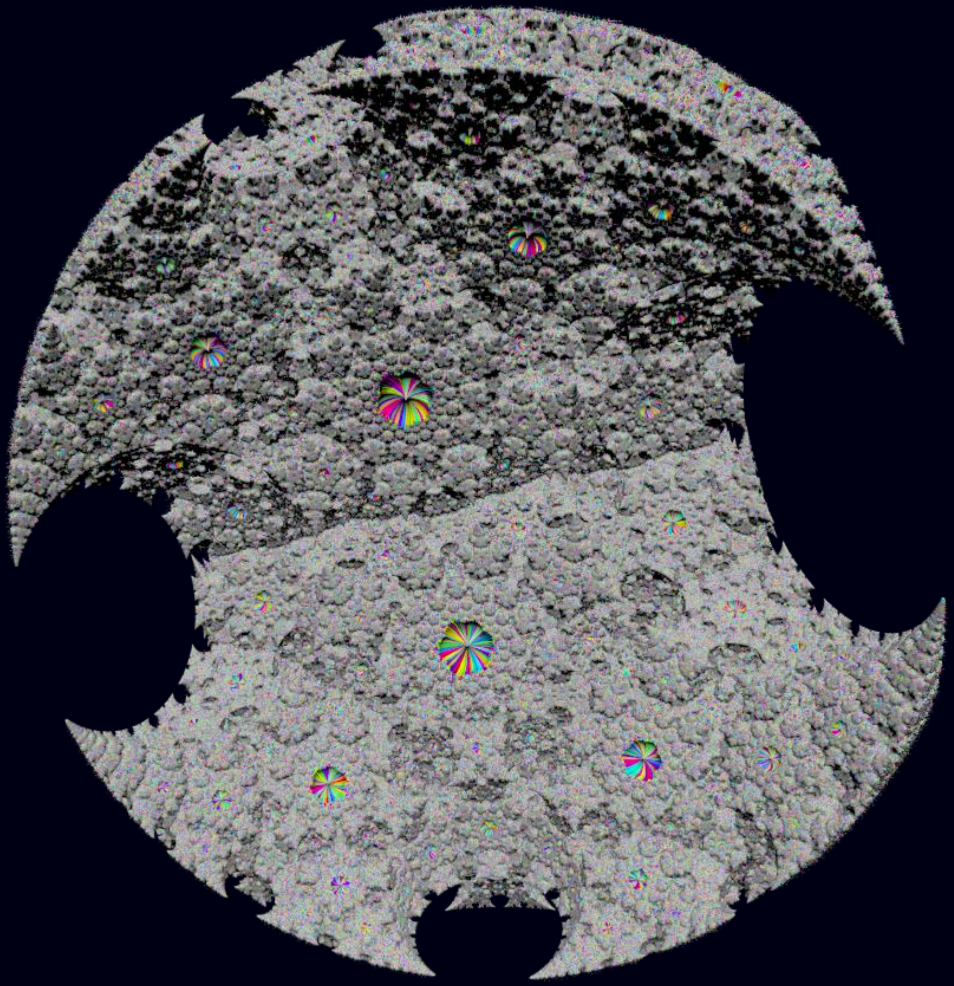


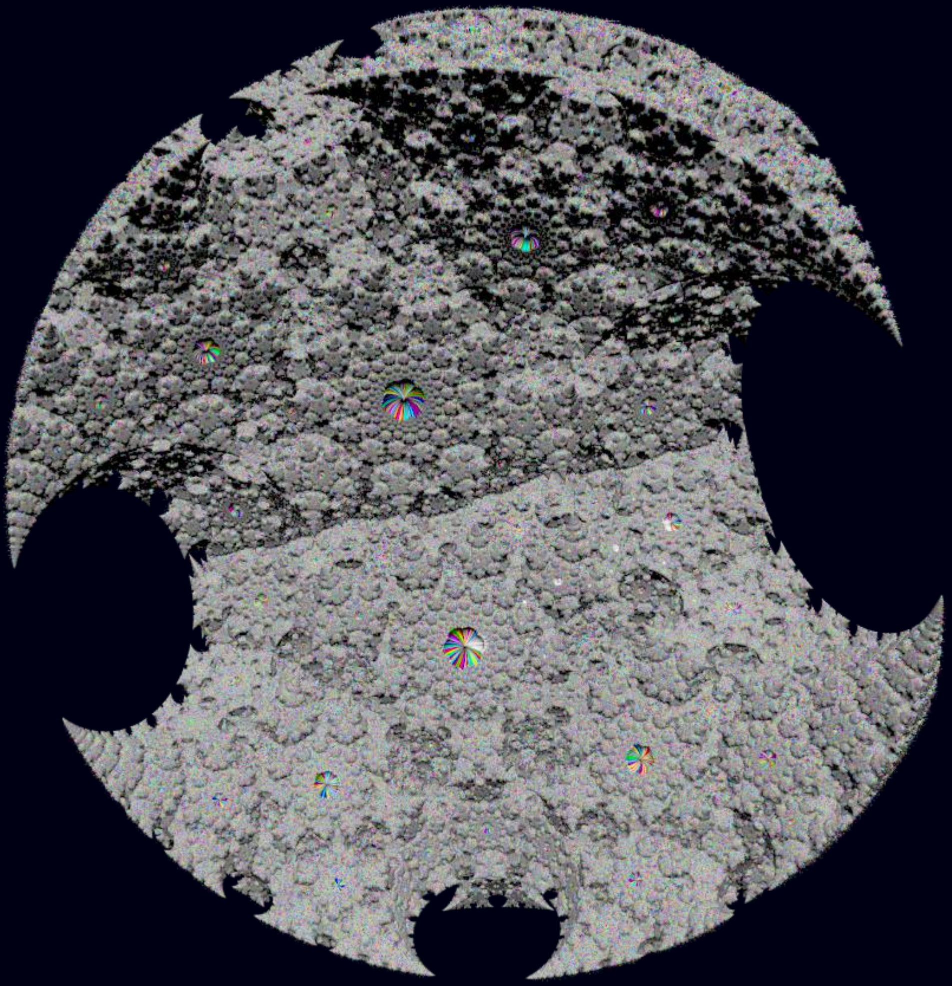
Tessellation by G



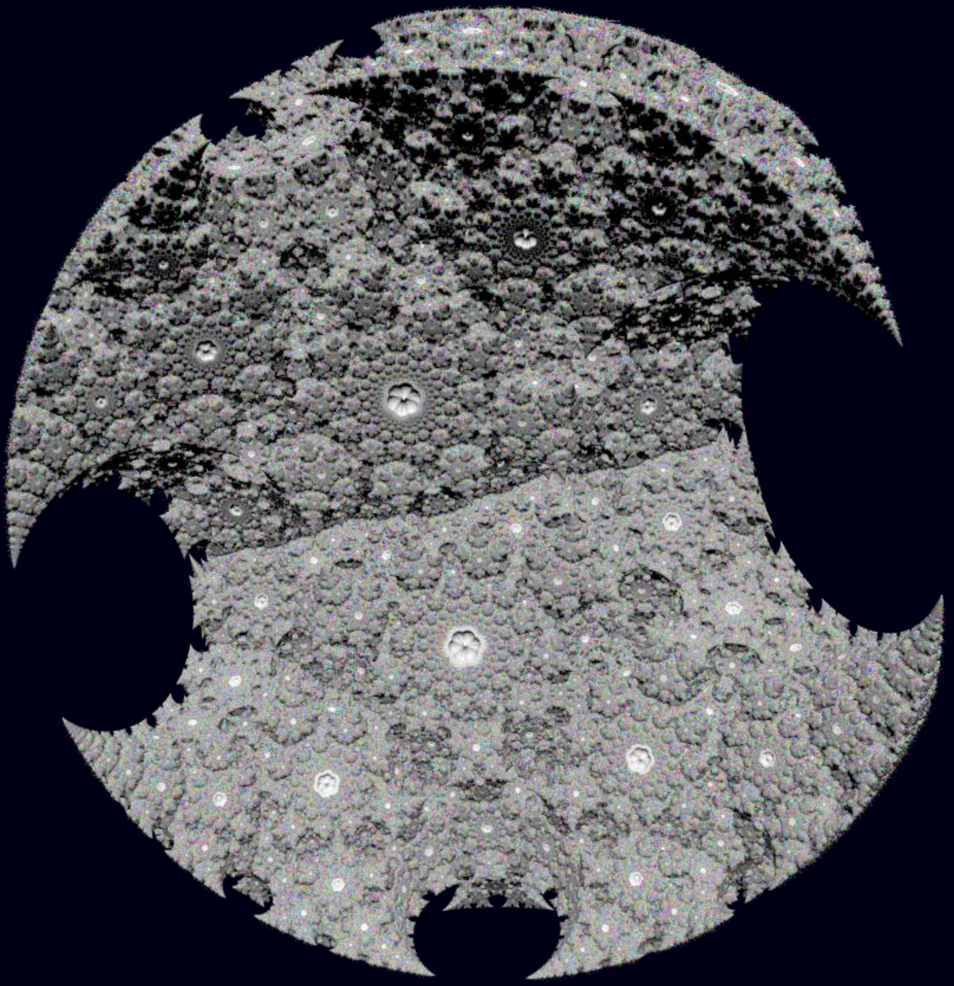
Tessellation by G



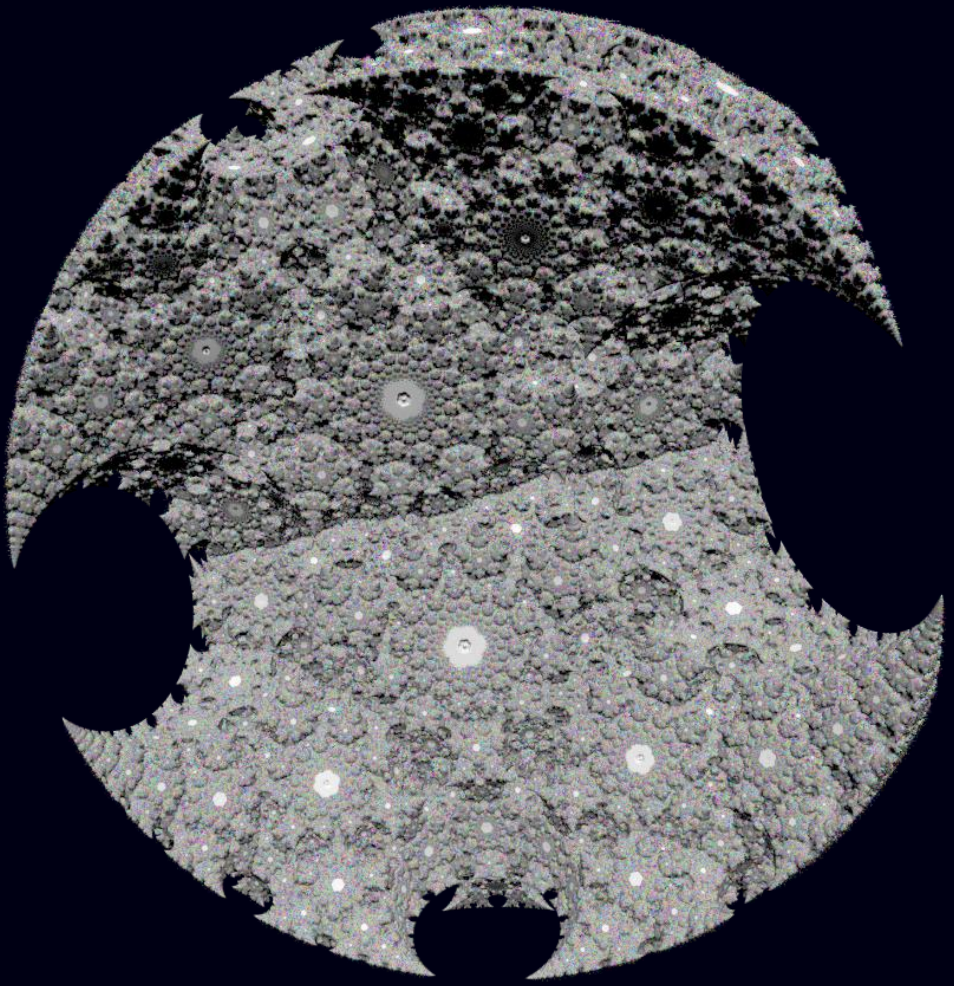
Tessellation by G



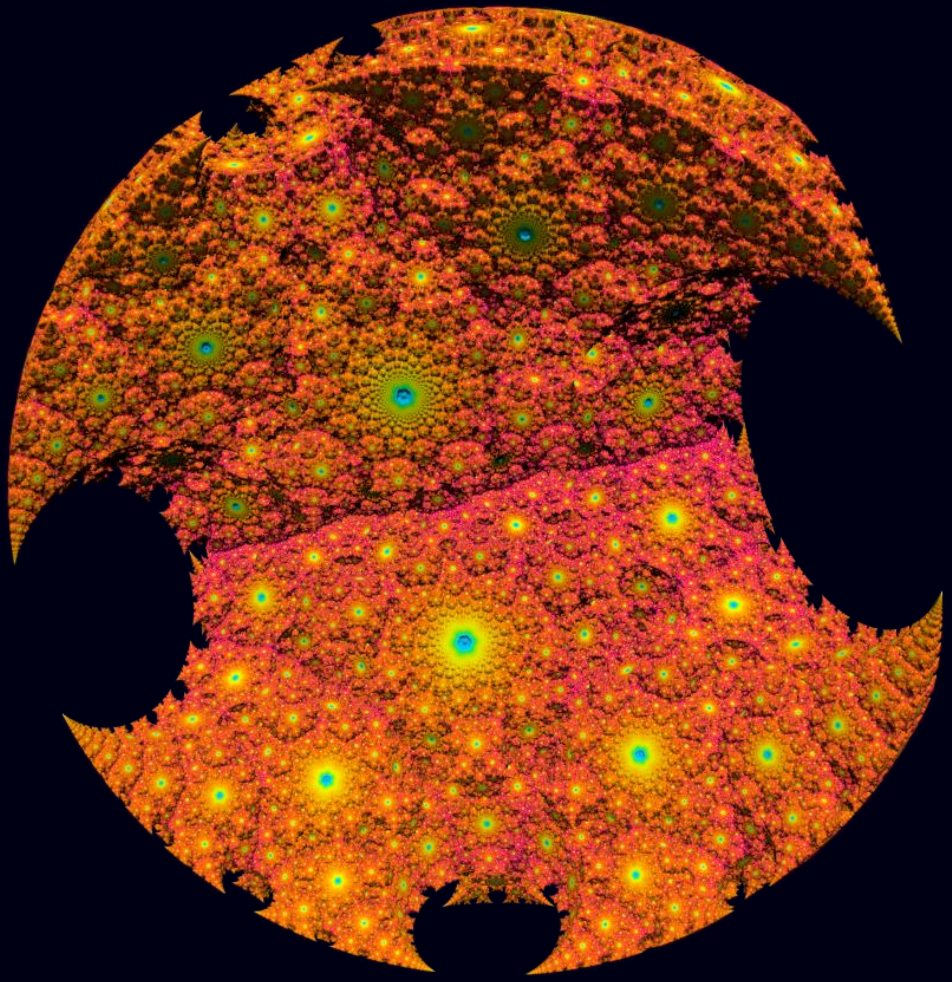
Tessellation by G



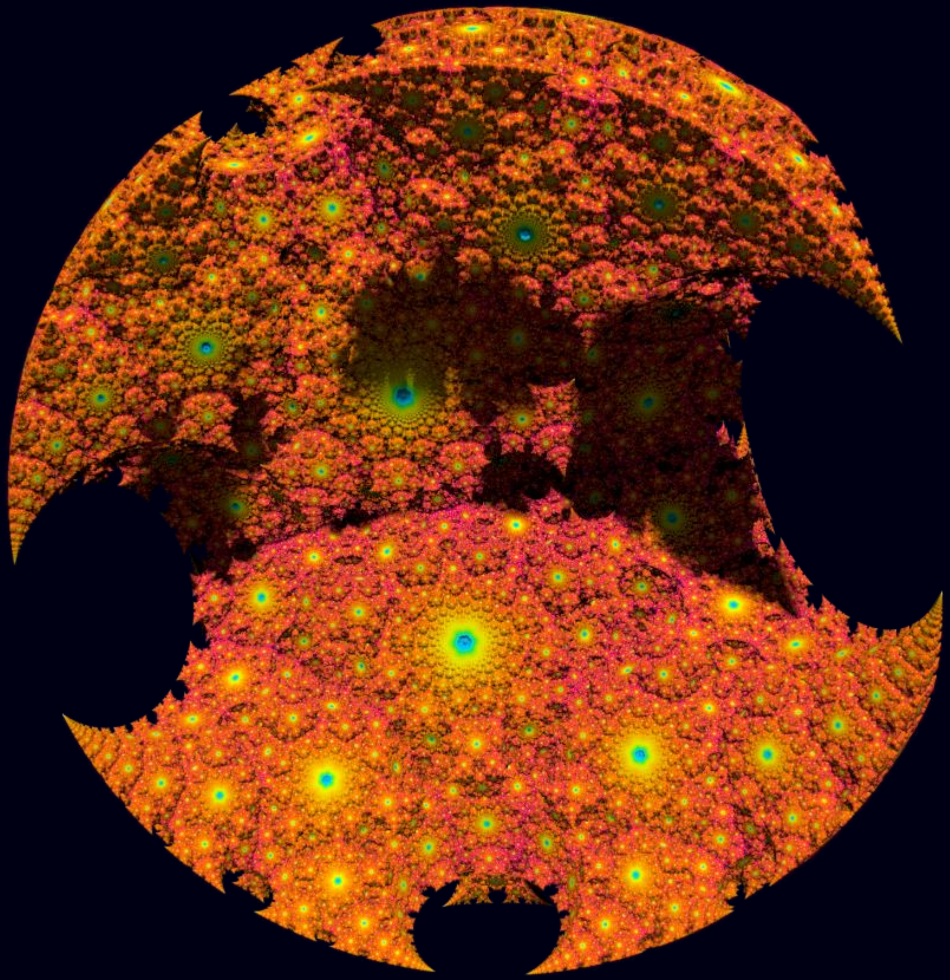
Tessellation by G



Tessellation by G



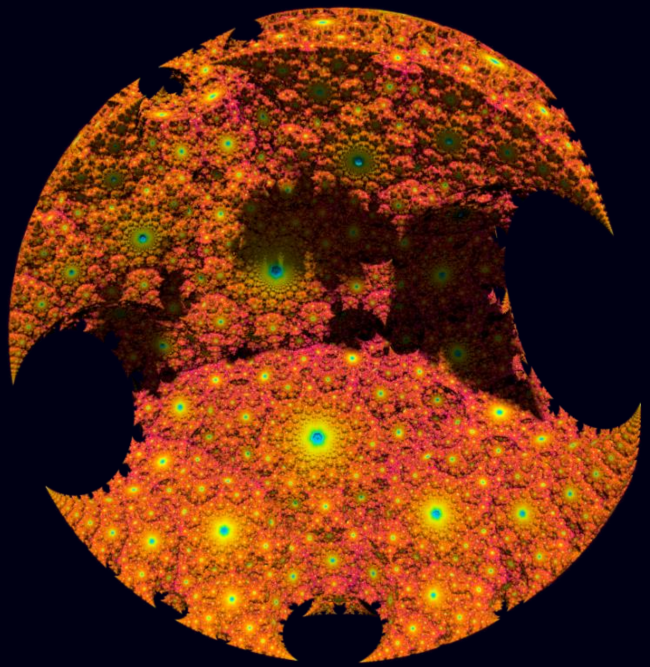
The Limit Set of G



Rationality and Ideality

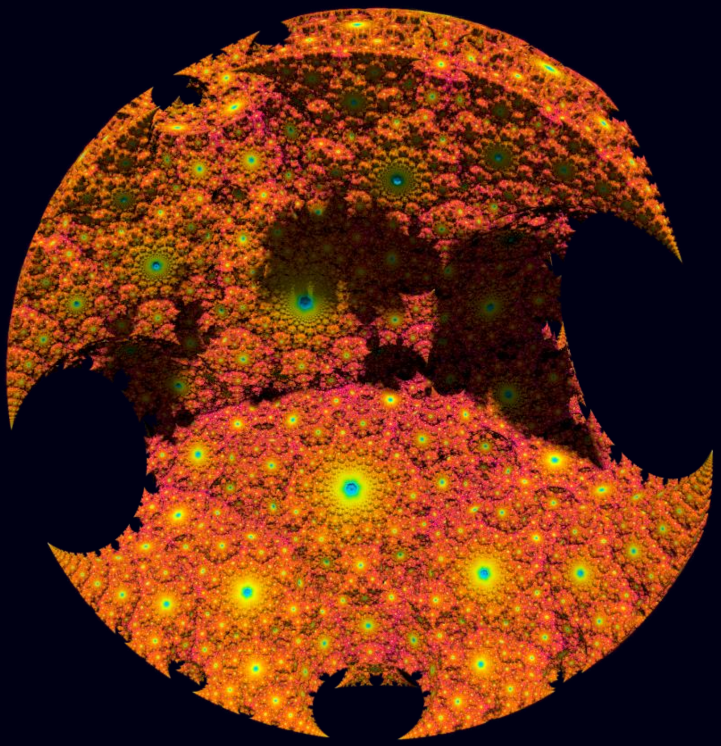
Two properties to characterize sphairahedron

If a sphairahedron is rational and ideal, G is discrete.

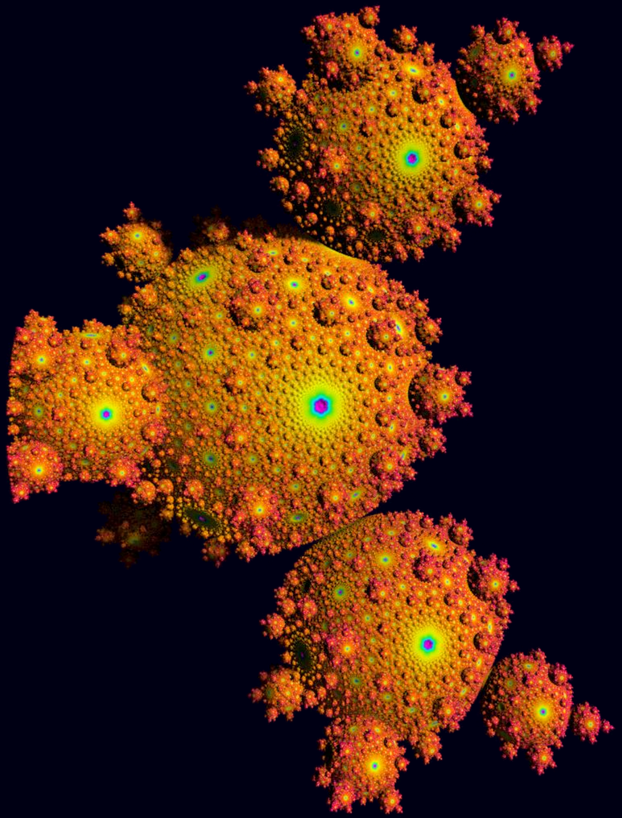


Rational Ideal Sphairahedron Group

Sphairahedron



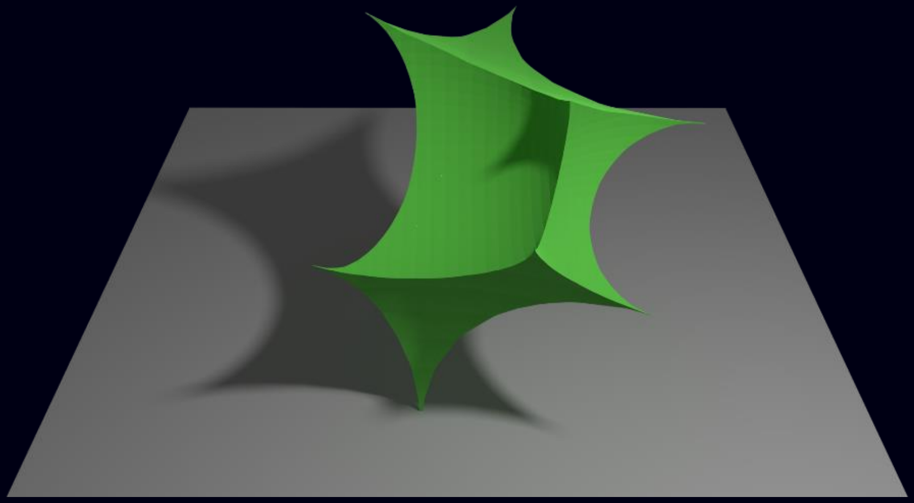
Semi-sphairahedron



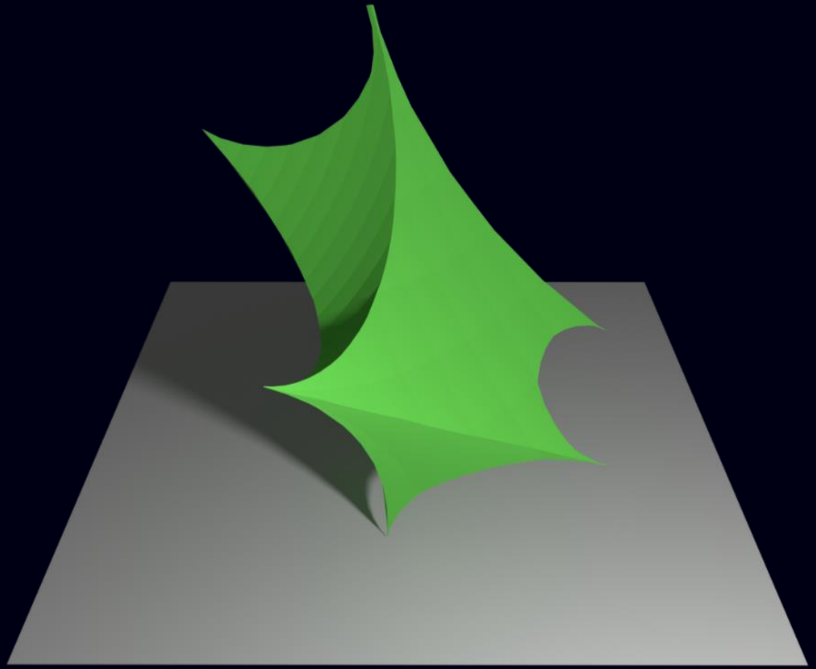
Quasi-sphere (homeomorphic to a sphere)

Rationality (Regularity)

All of the dihedral angles of edges is rational.
(π/n for the natural number n)



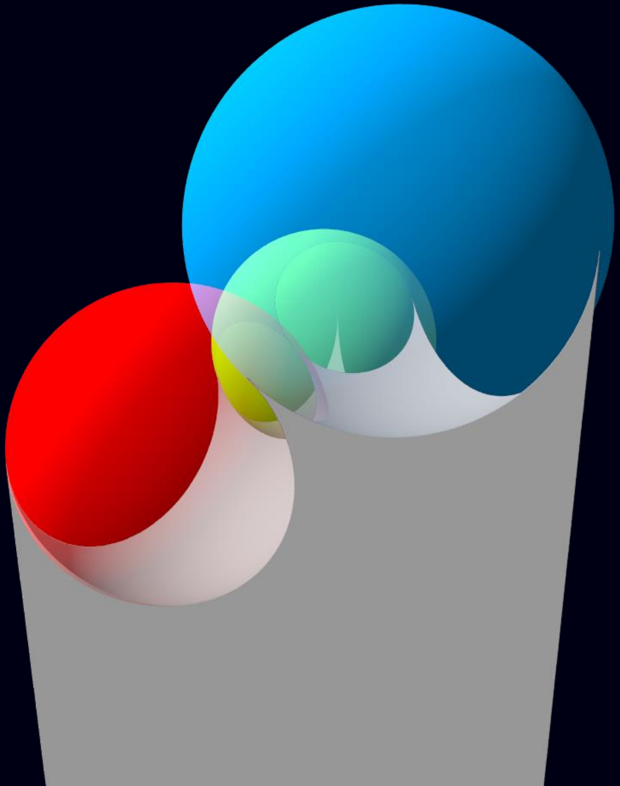
$\pi/3$

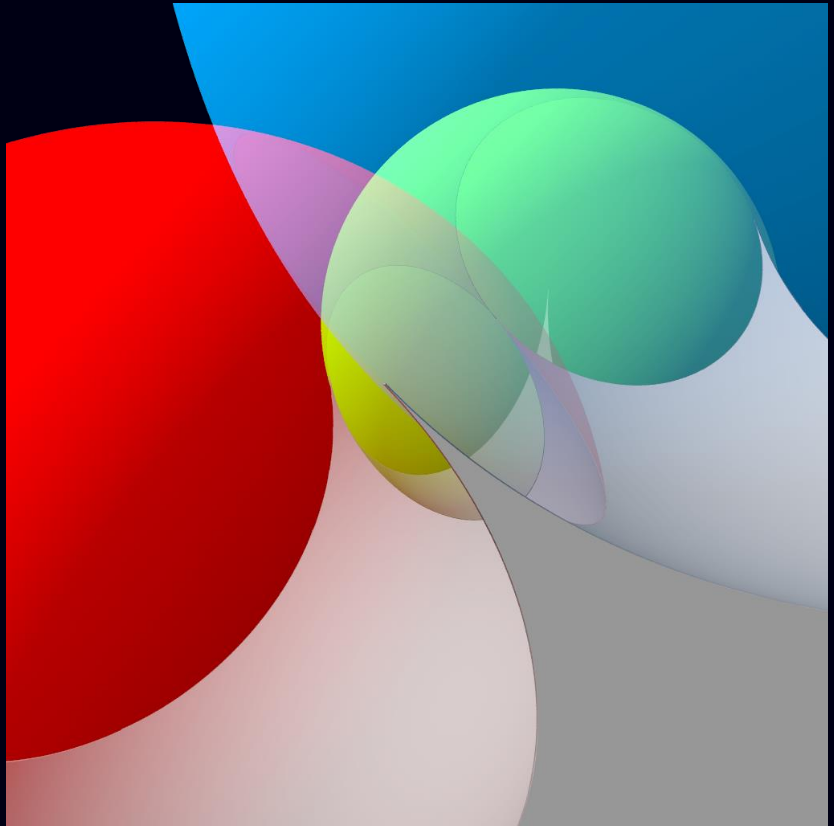


$\pi/2, \pi/3, \pi/6$

Ideality

All of the edges are mutually tangent at its vertex

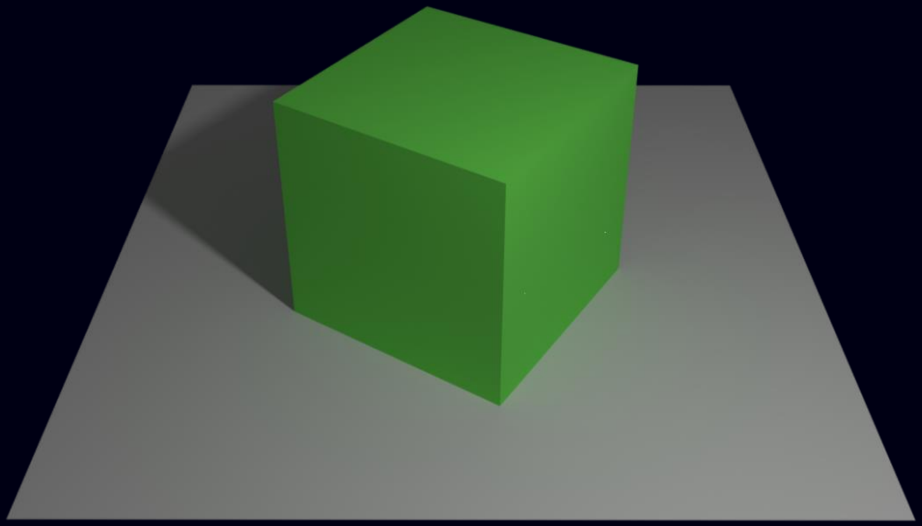


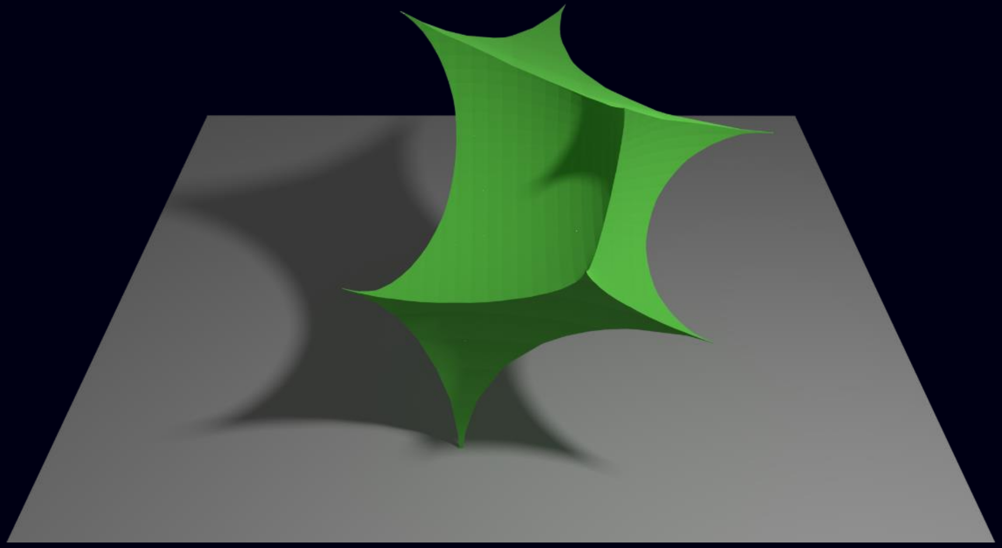


Parameter Space

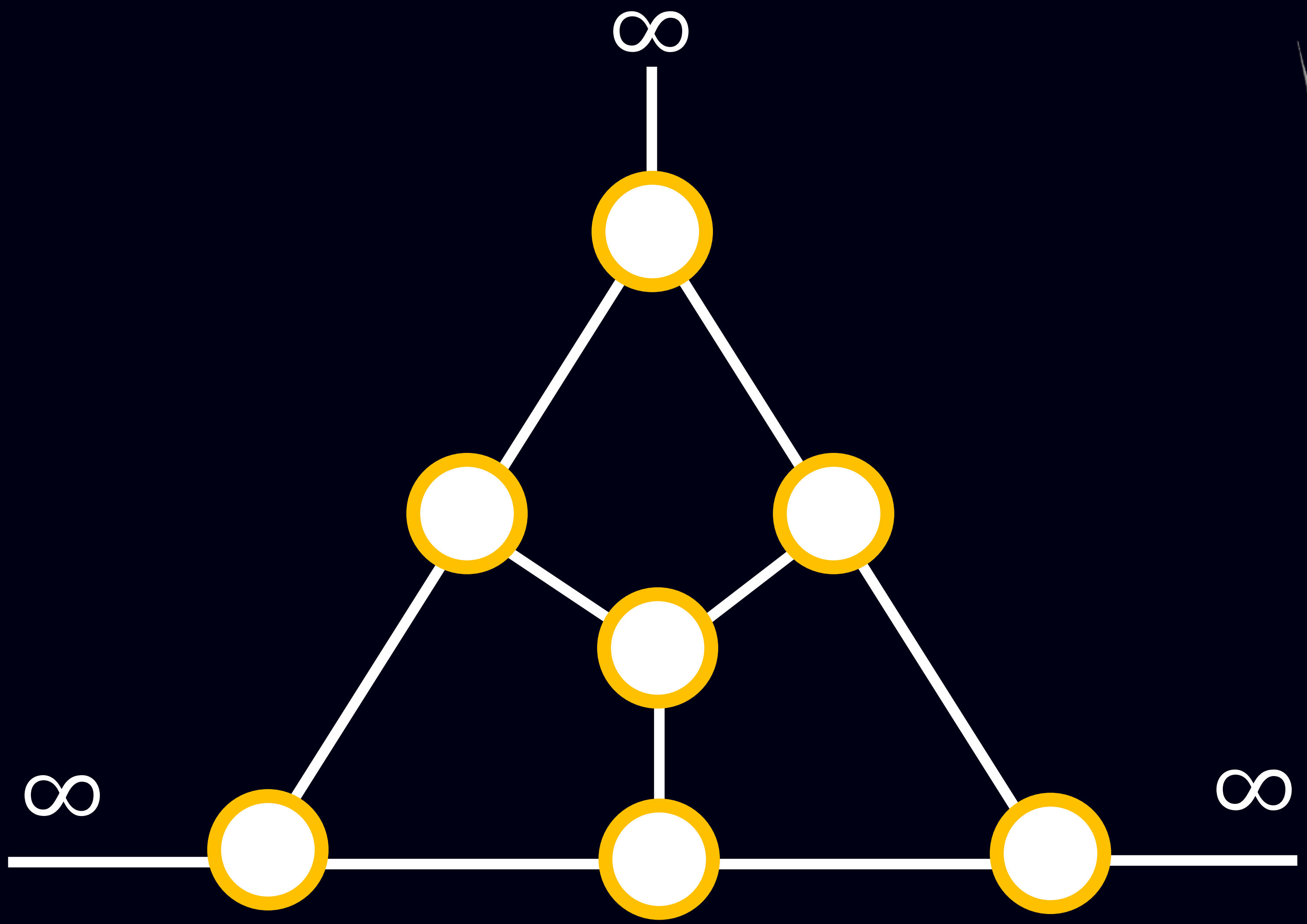
Derivation of Parameter Space

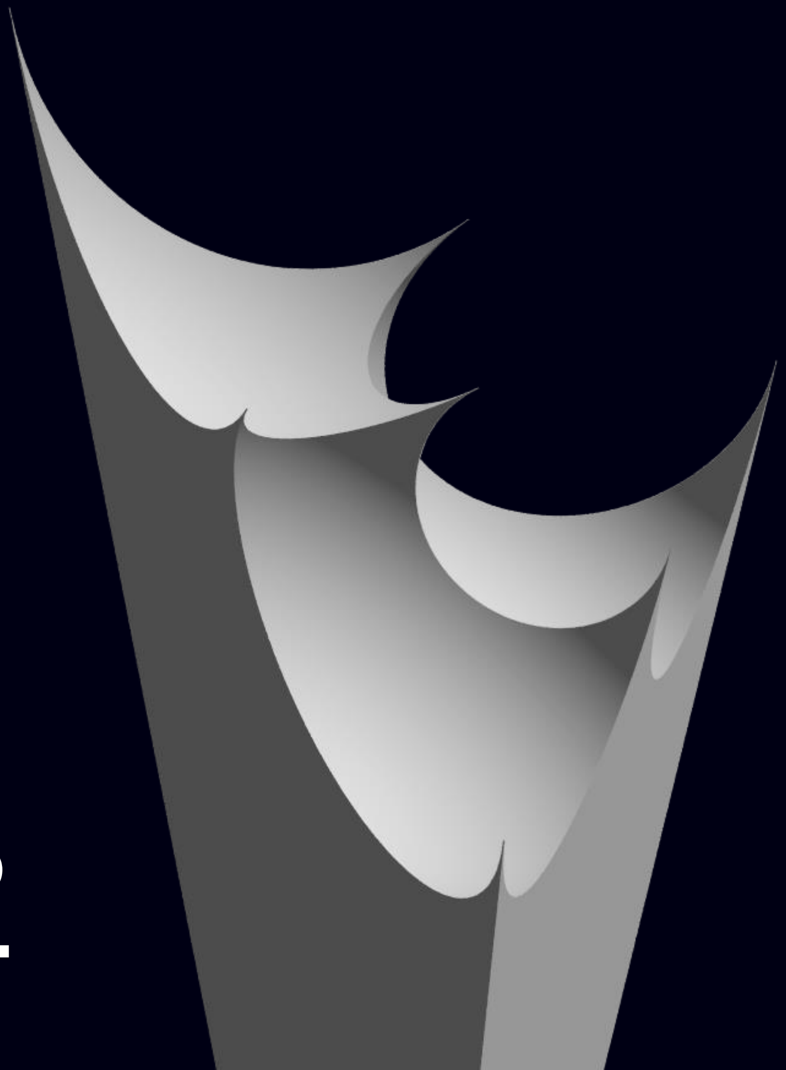
Cube-type sphairahedron



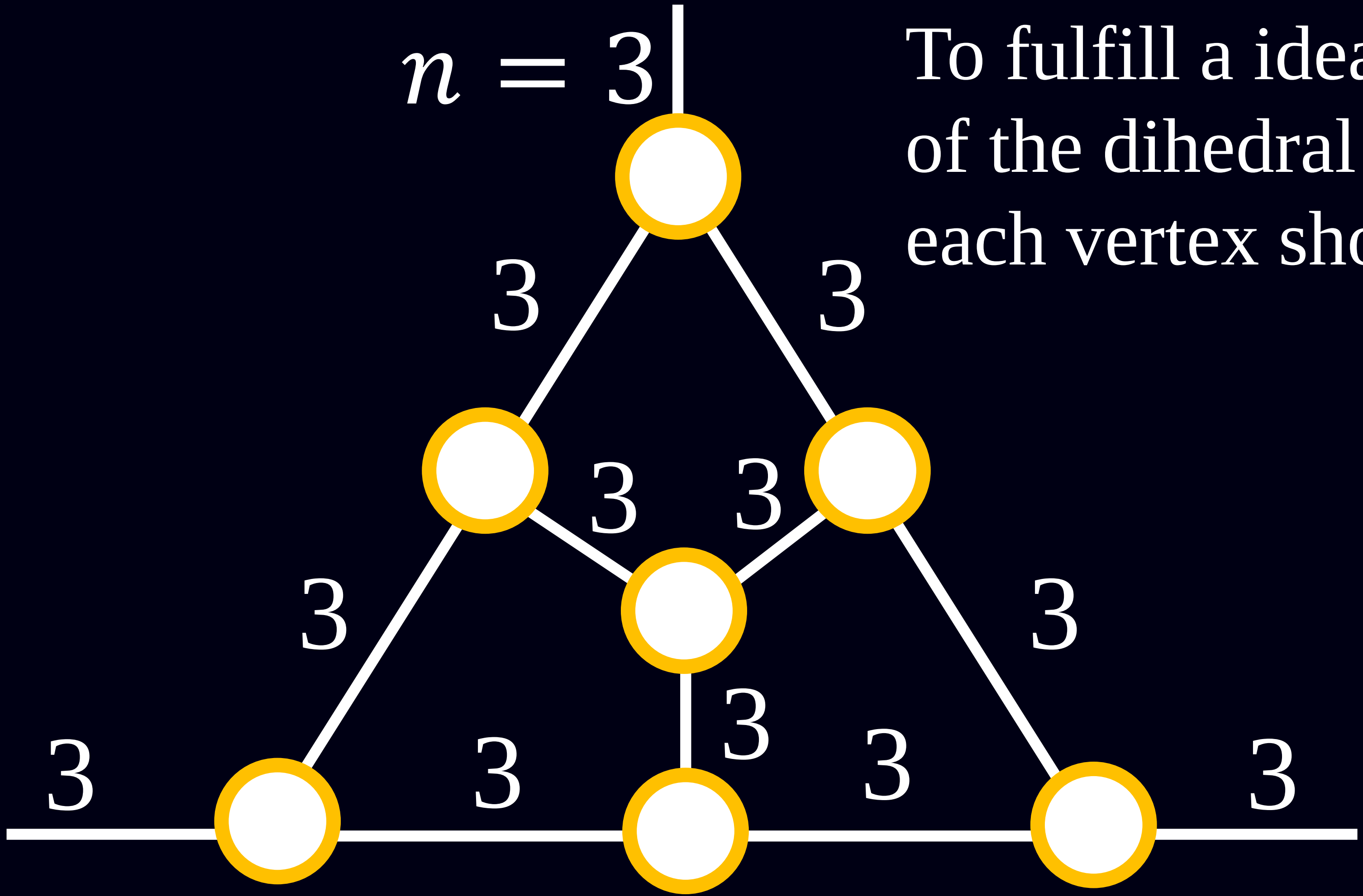


Graph Representation



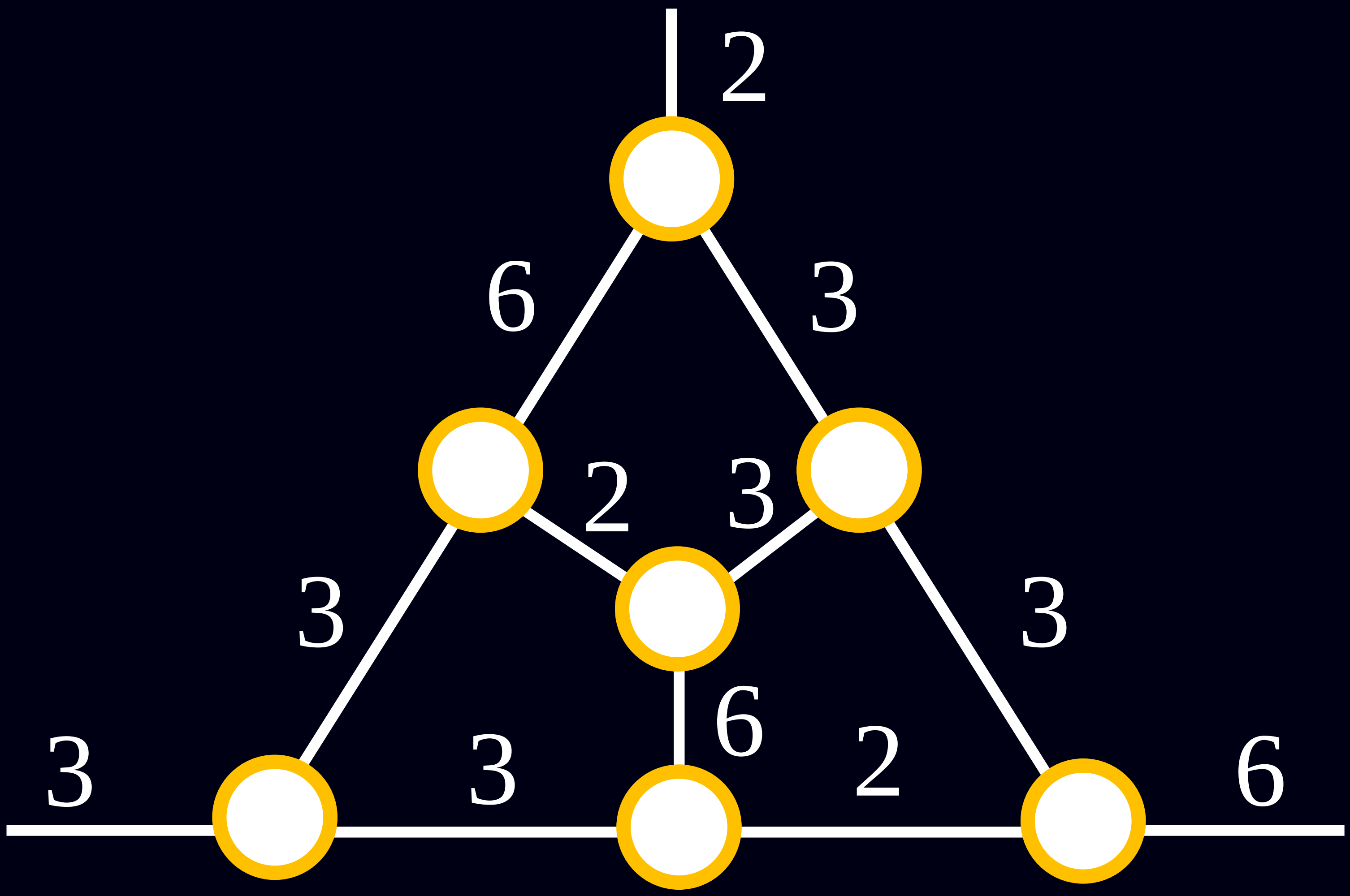


Combination of Dihedral Angles

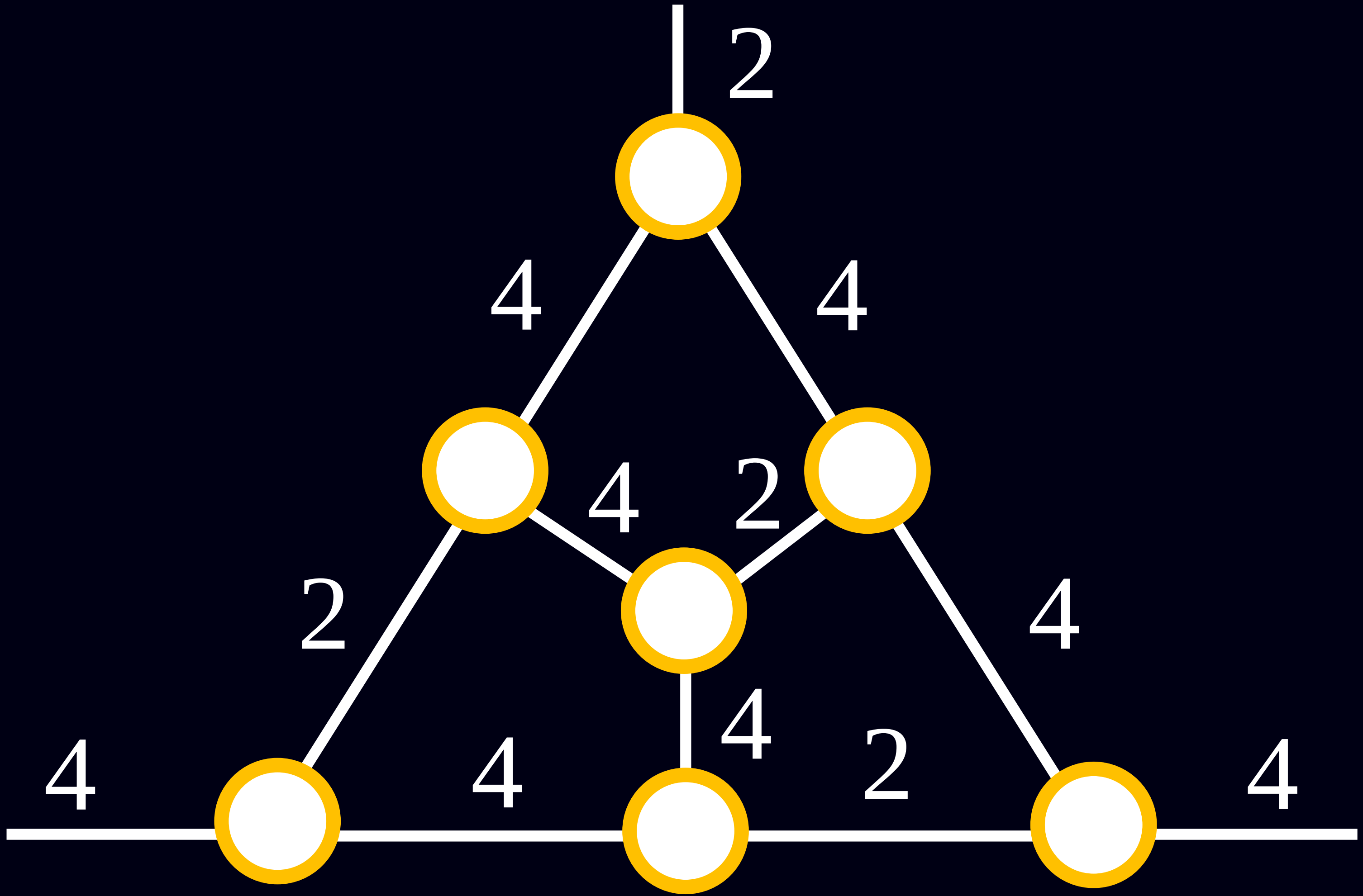


To fulfill a ideality, the sum of the dihedral angles at each vertex should be π

Combination of Dihedral Angles



Combination of Dihedral Angles



Derivation of Parameter Space

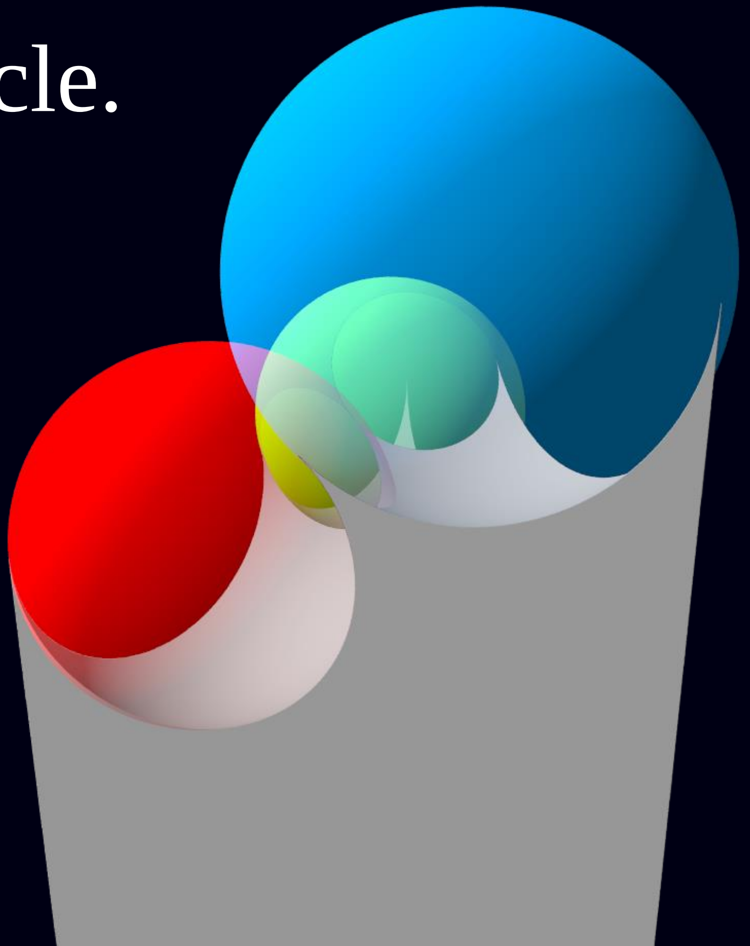
Fix prism and a sphere

- The prism is inscribed inside an unit circle.
- The height of the red sphere is 0.

Parameter

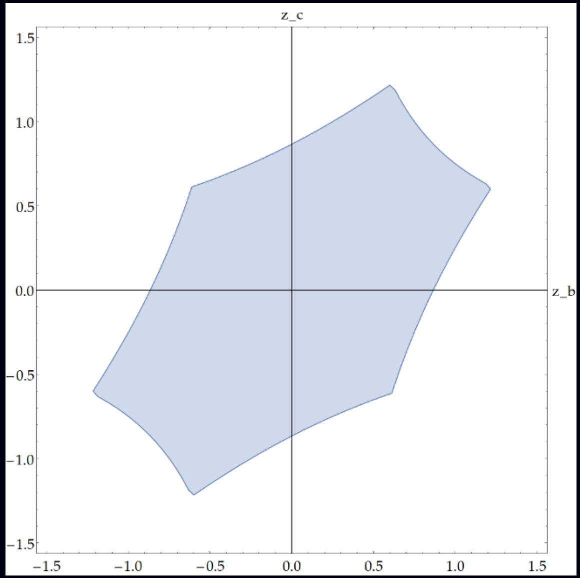
z_b : The height of the green sphere

z_c : The height of the blue sphere



Derivation of Parameter Space

All of the dihedral angles are $\pi/3$



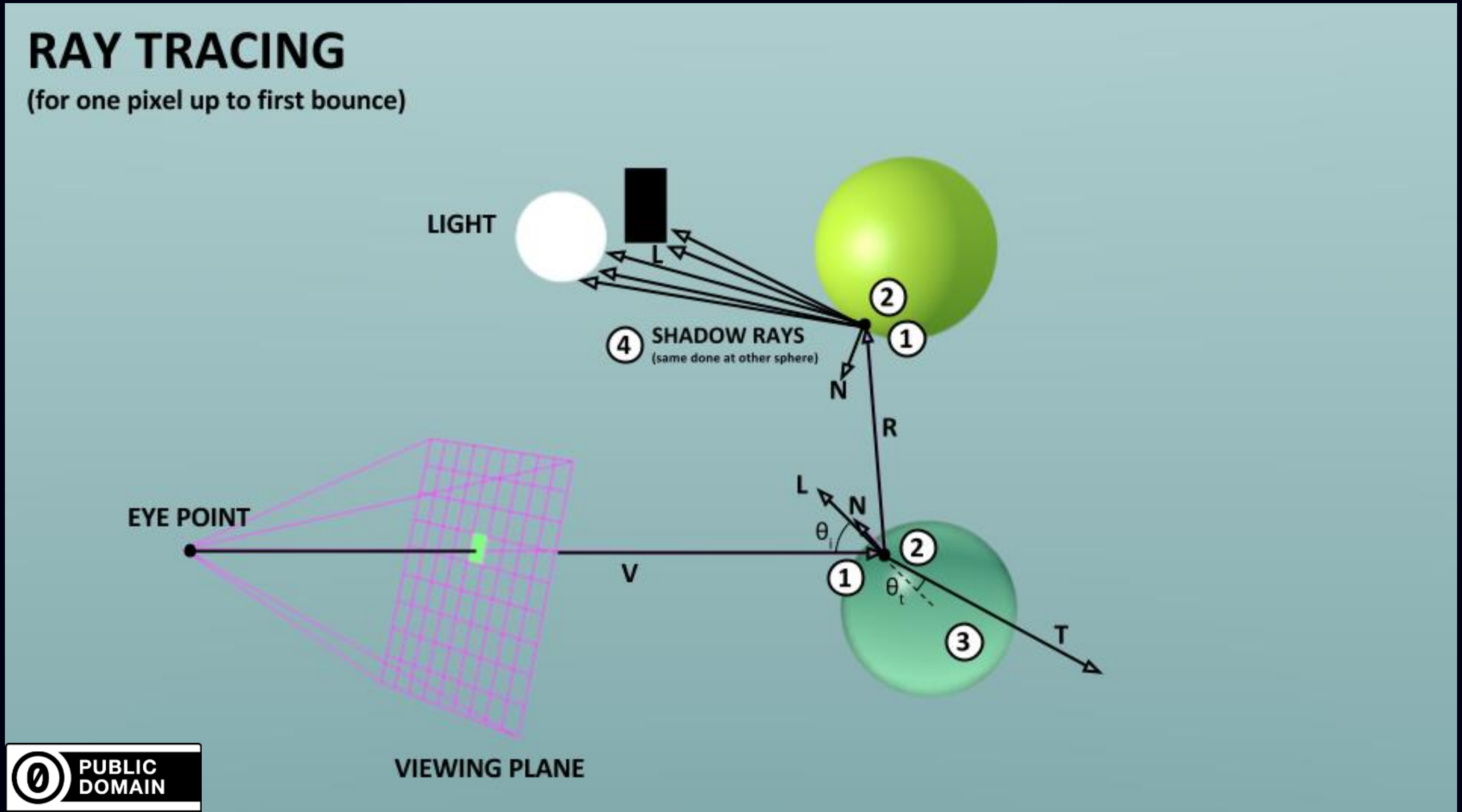
$$\begin{cases} z_b z_c < 3/4 \\ z_c^2 - z_b z_c < 3/4 \\ z_b^2 - z_b z_c < 3/4 \end{cases}$$

Parameter space of the cube-type sphairahedron is studied by Ahara and Araki (2003) and also Ryo Kageyama (2016).

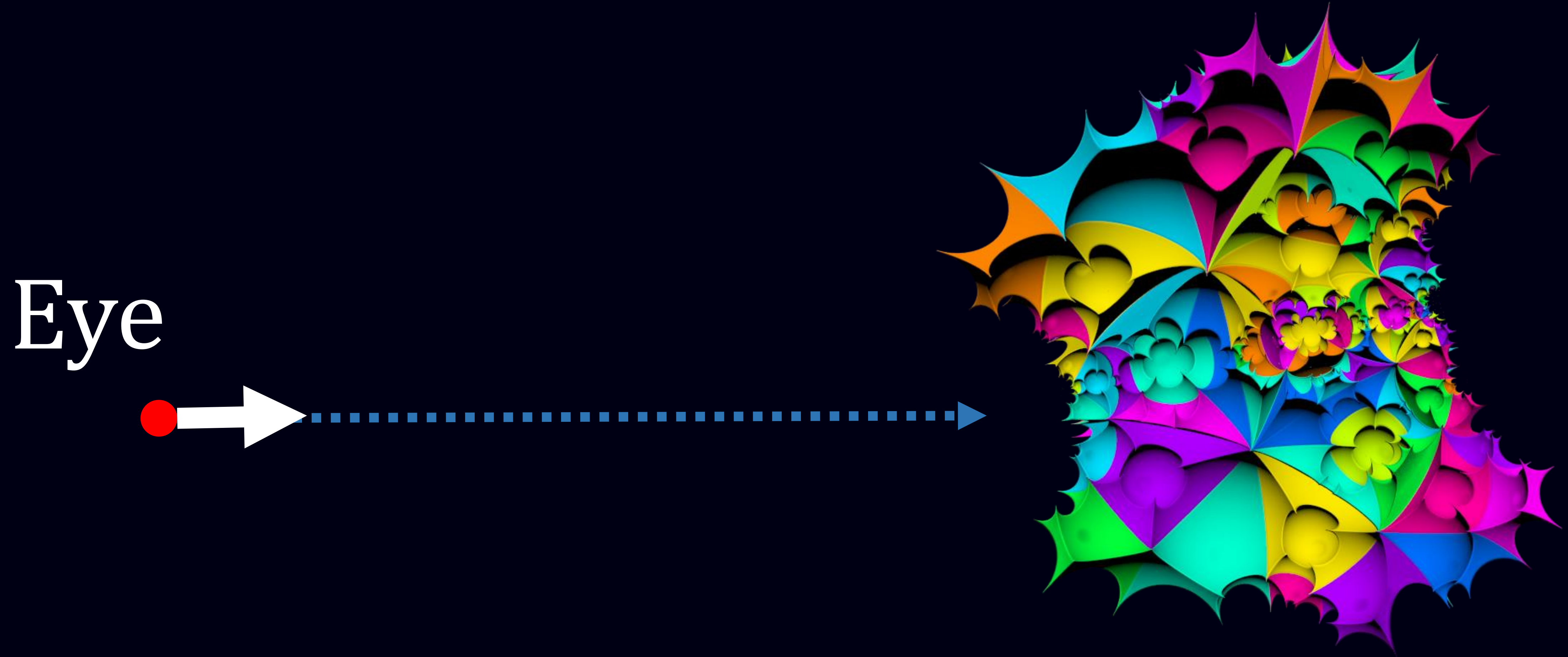
Rendering Technique

Ray Tracing

Suited for parallel computing by GPU



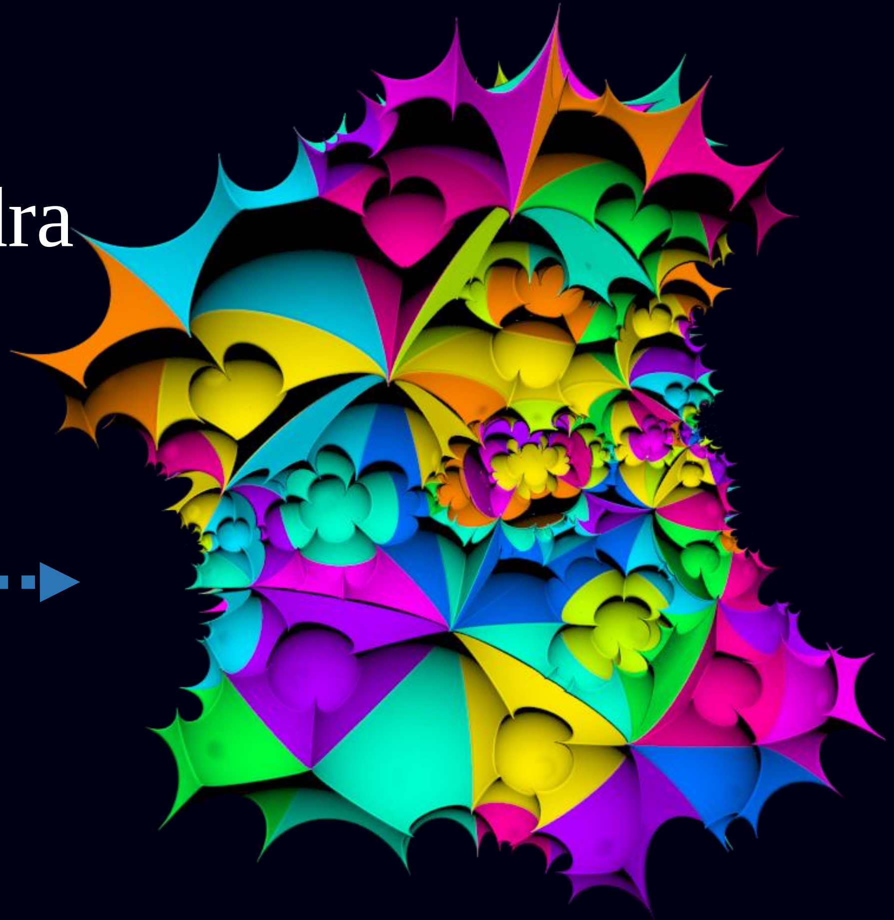
Ray Tracing



Ray Tracing

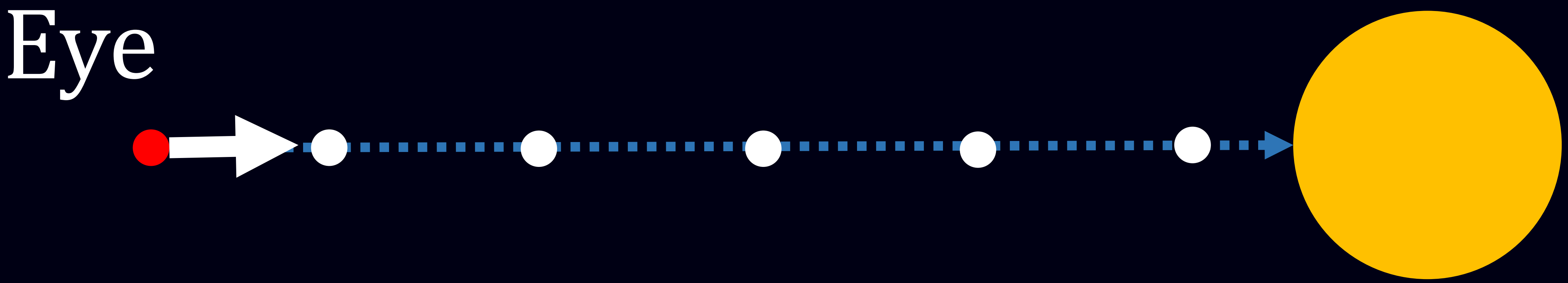
We have to compute an intersection between the ray and many sphairahedra

Eye

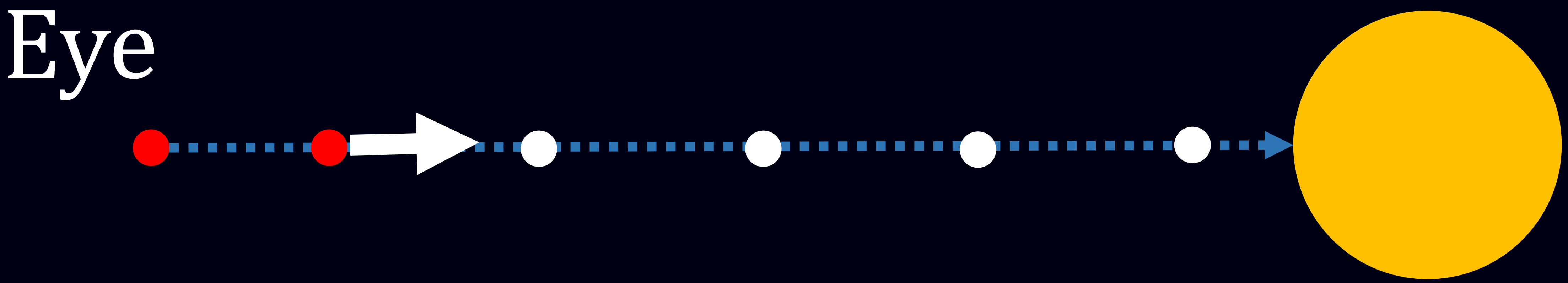


Ray Marching

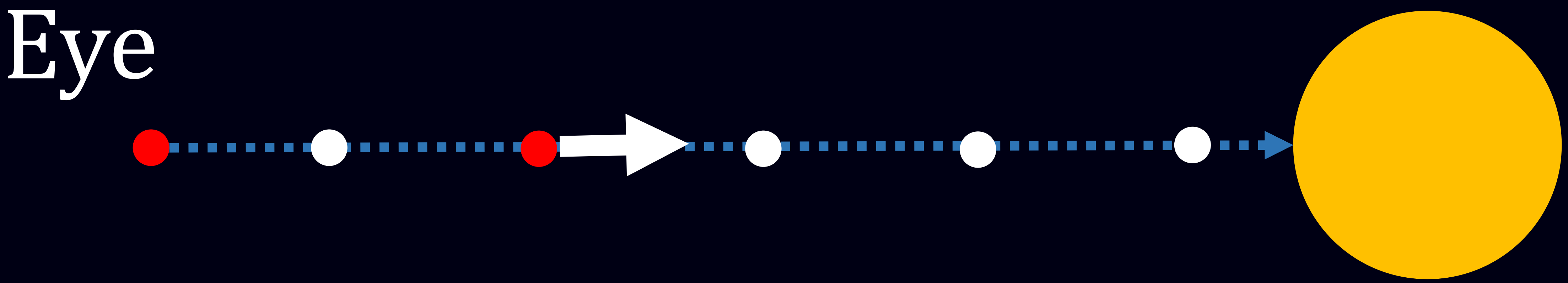
Find intersection between the ray and objects



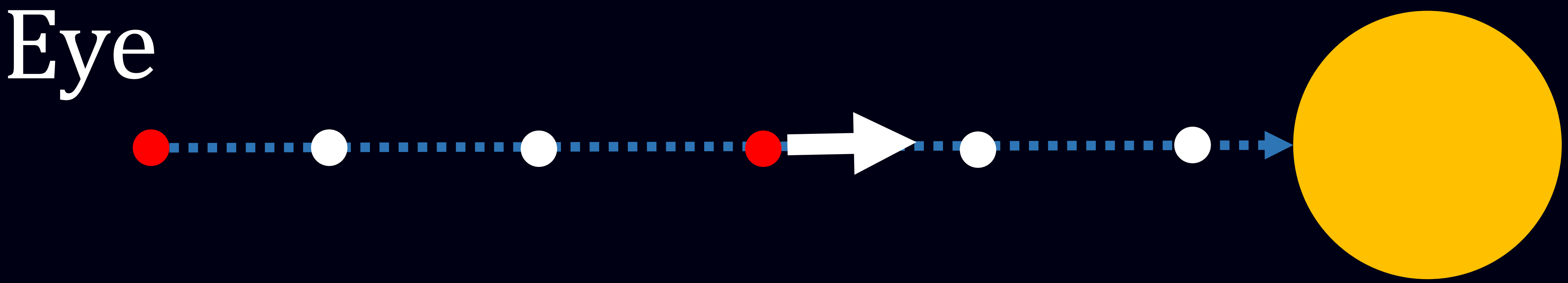
Ray Marching



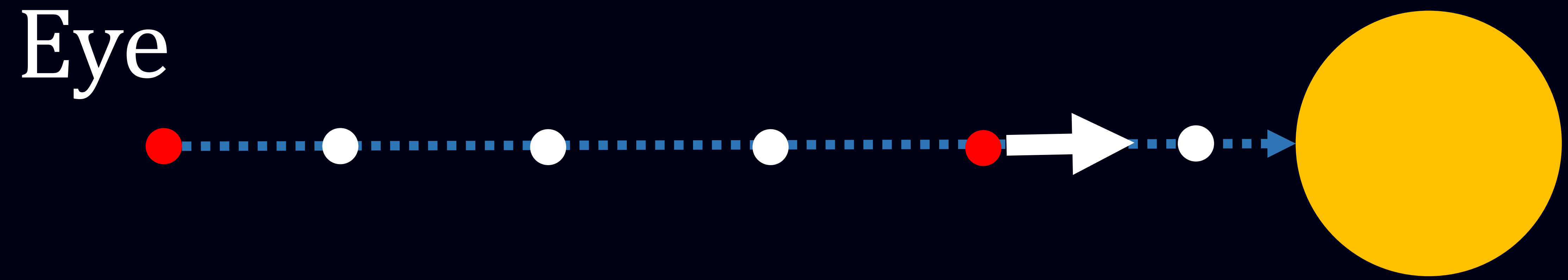
Ray Marching



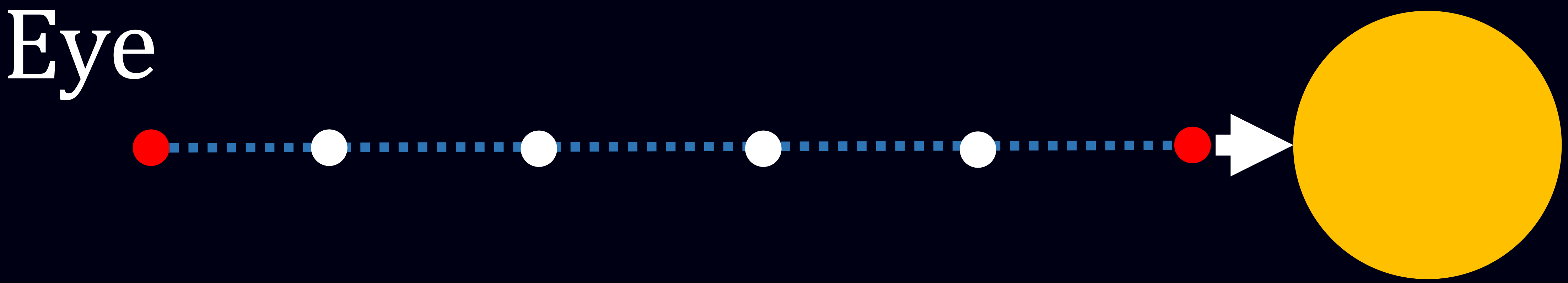
Ray Marching



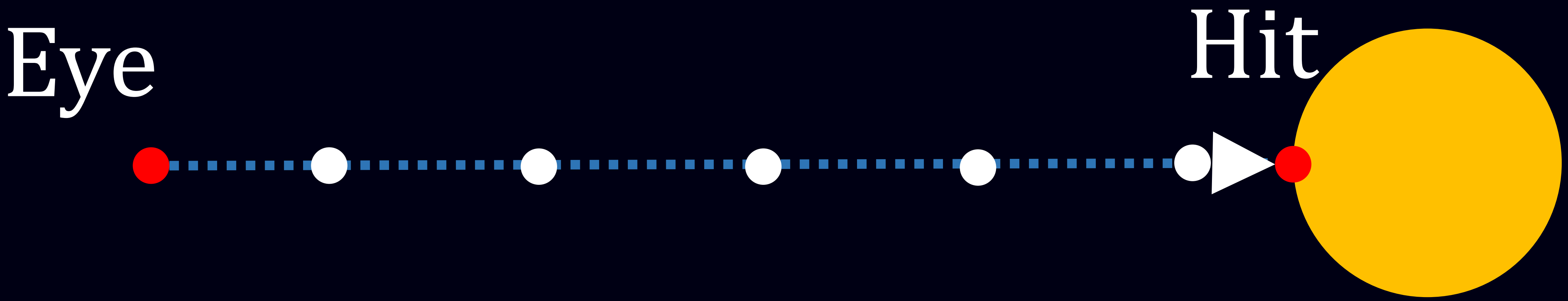
Ray Marching



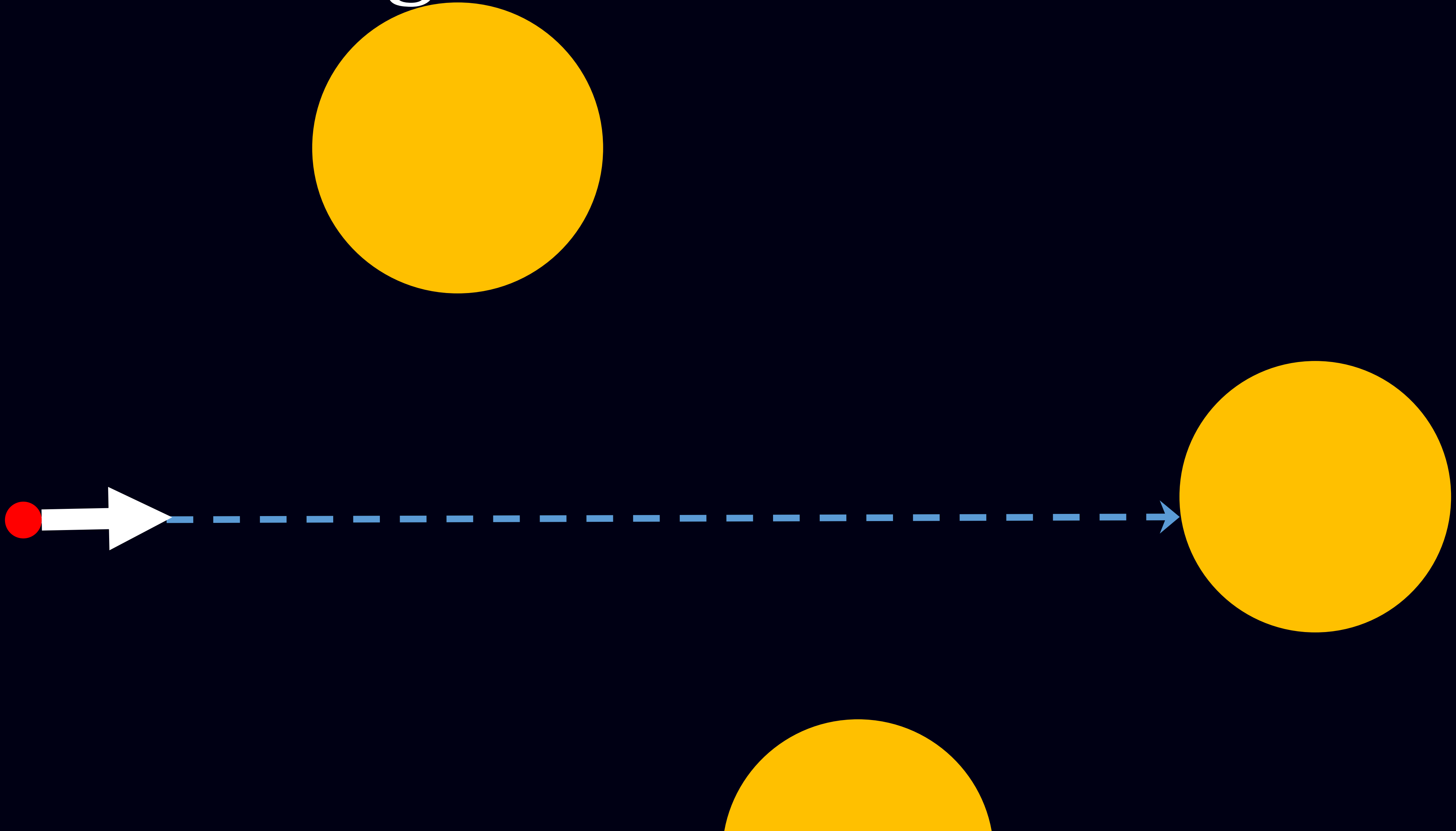
Ray Marching



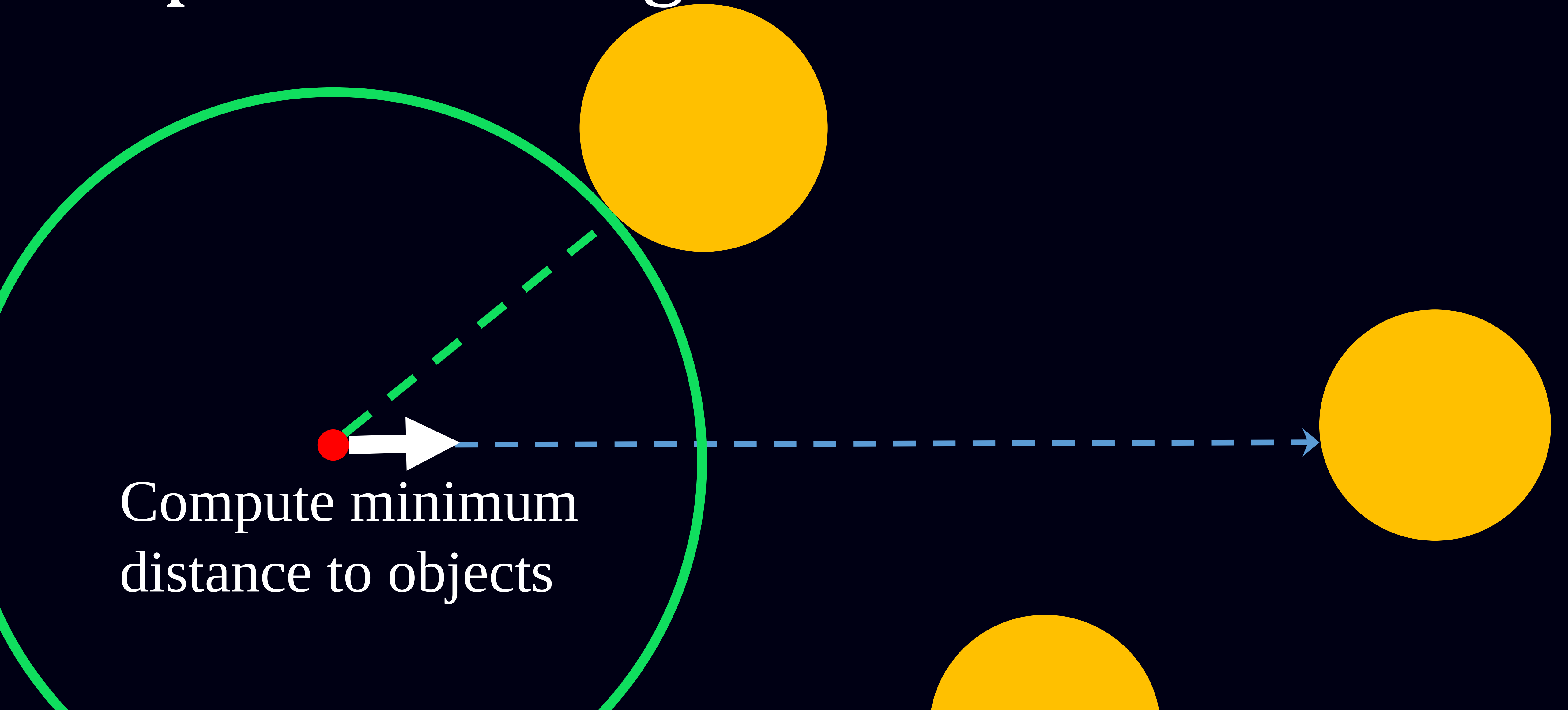
Ray Marching



Sphere Tracing

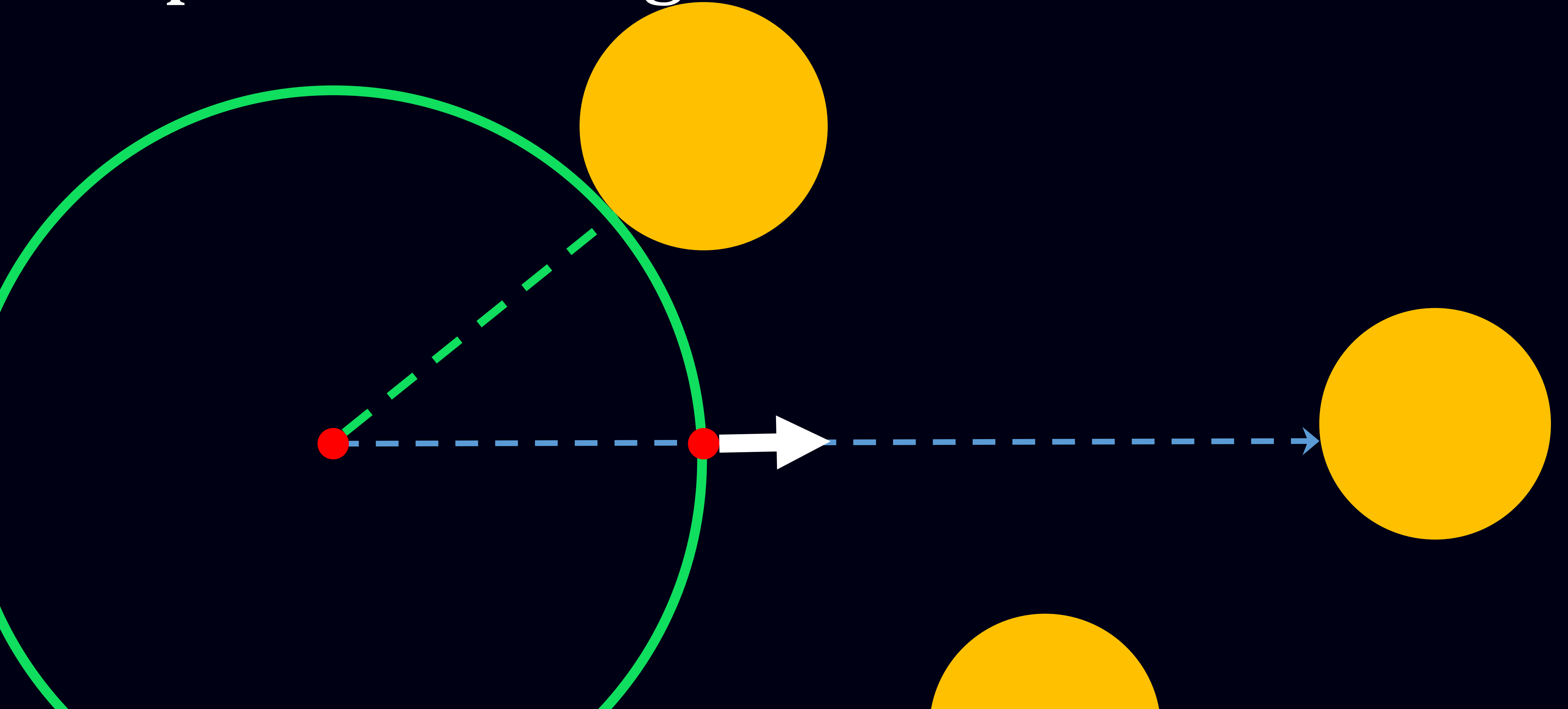


Sphere Tracing

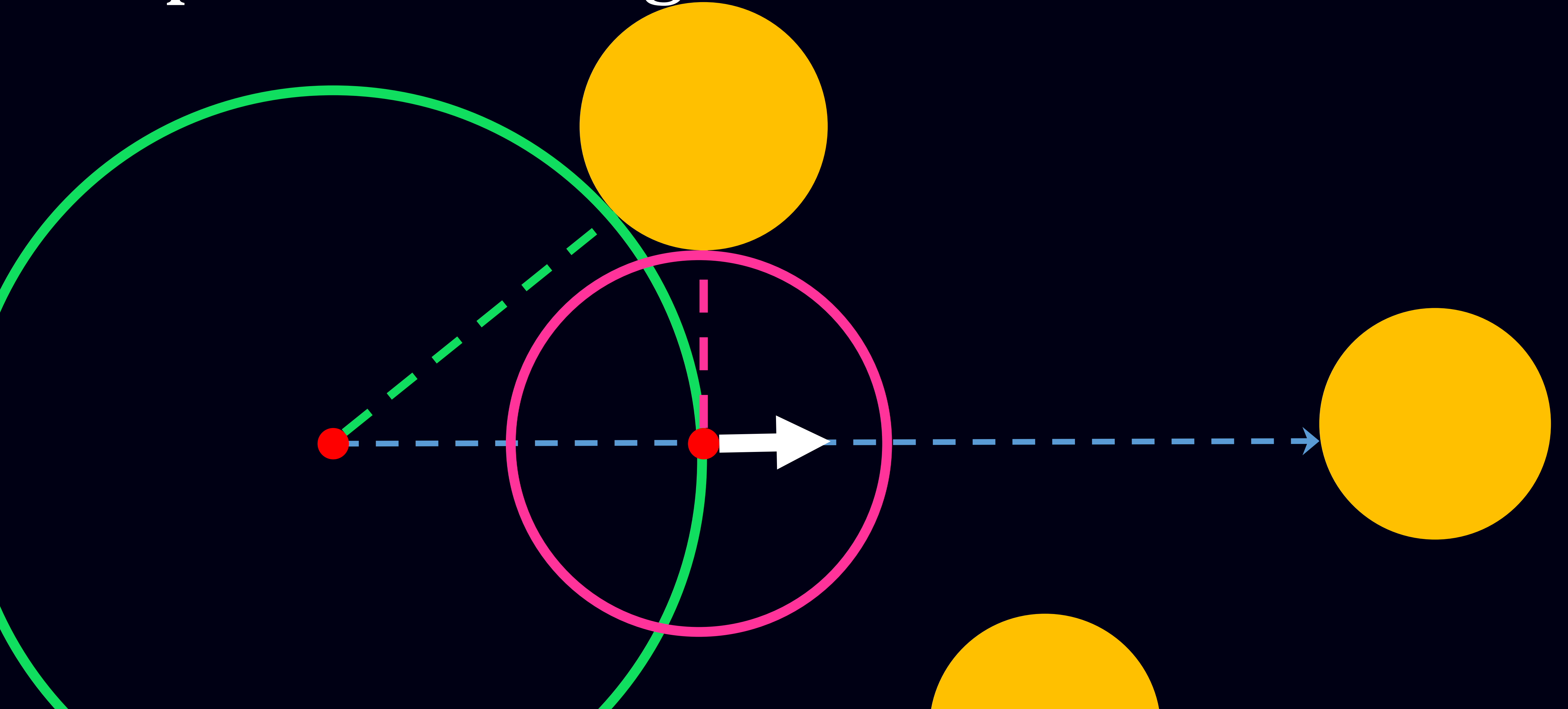


Compute minimum distance to objects

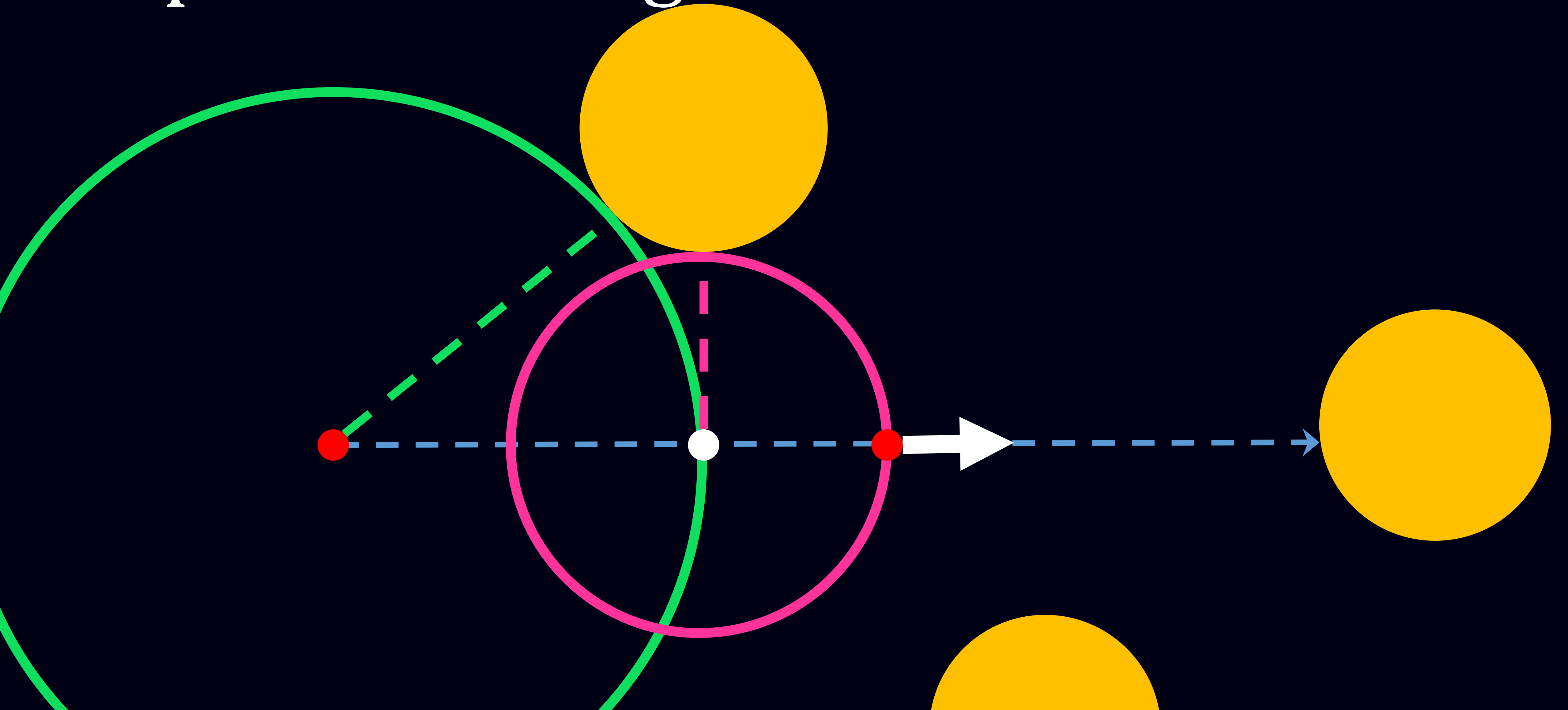
Sphere Tracing



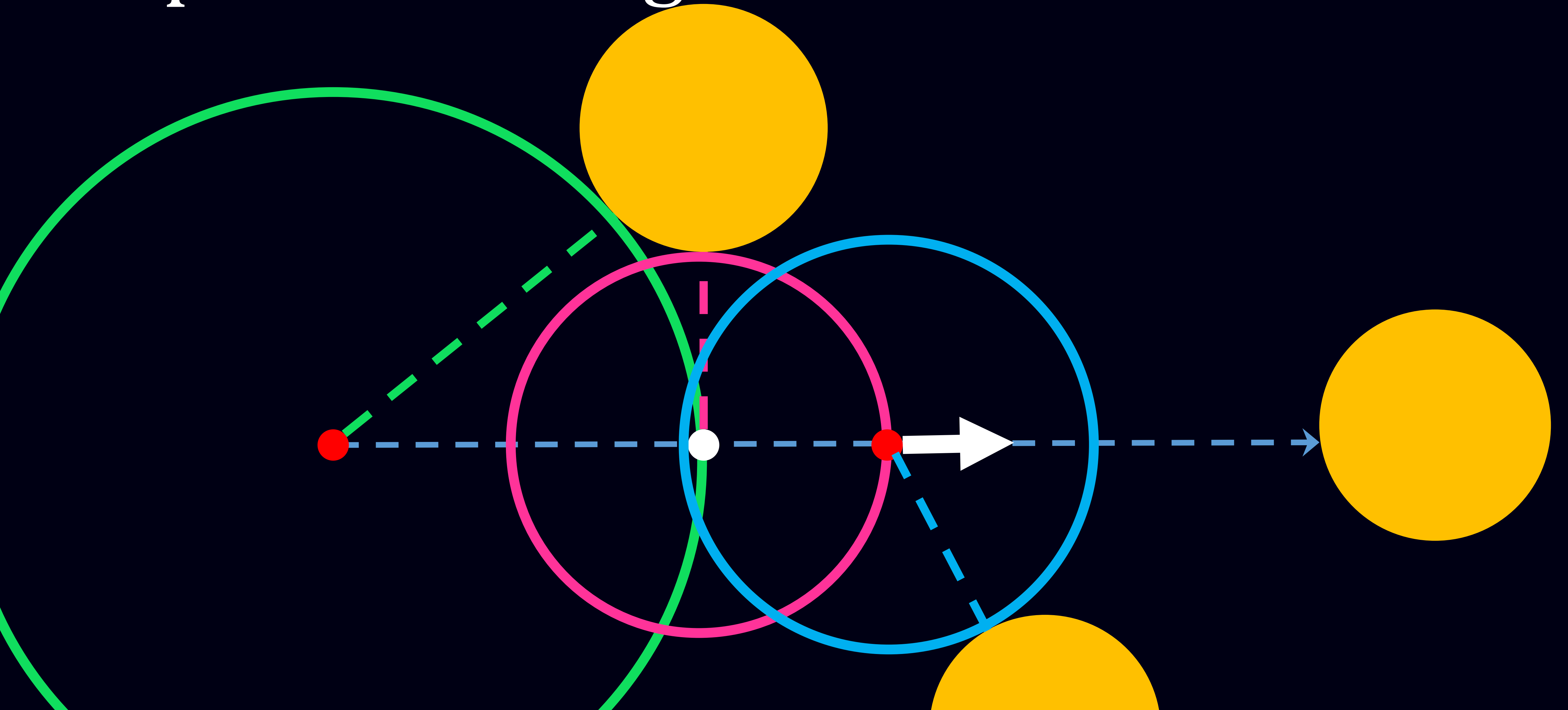
Sphere Tracing



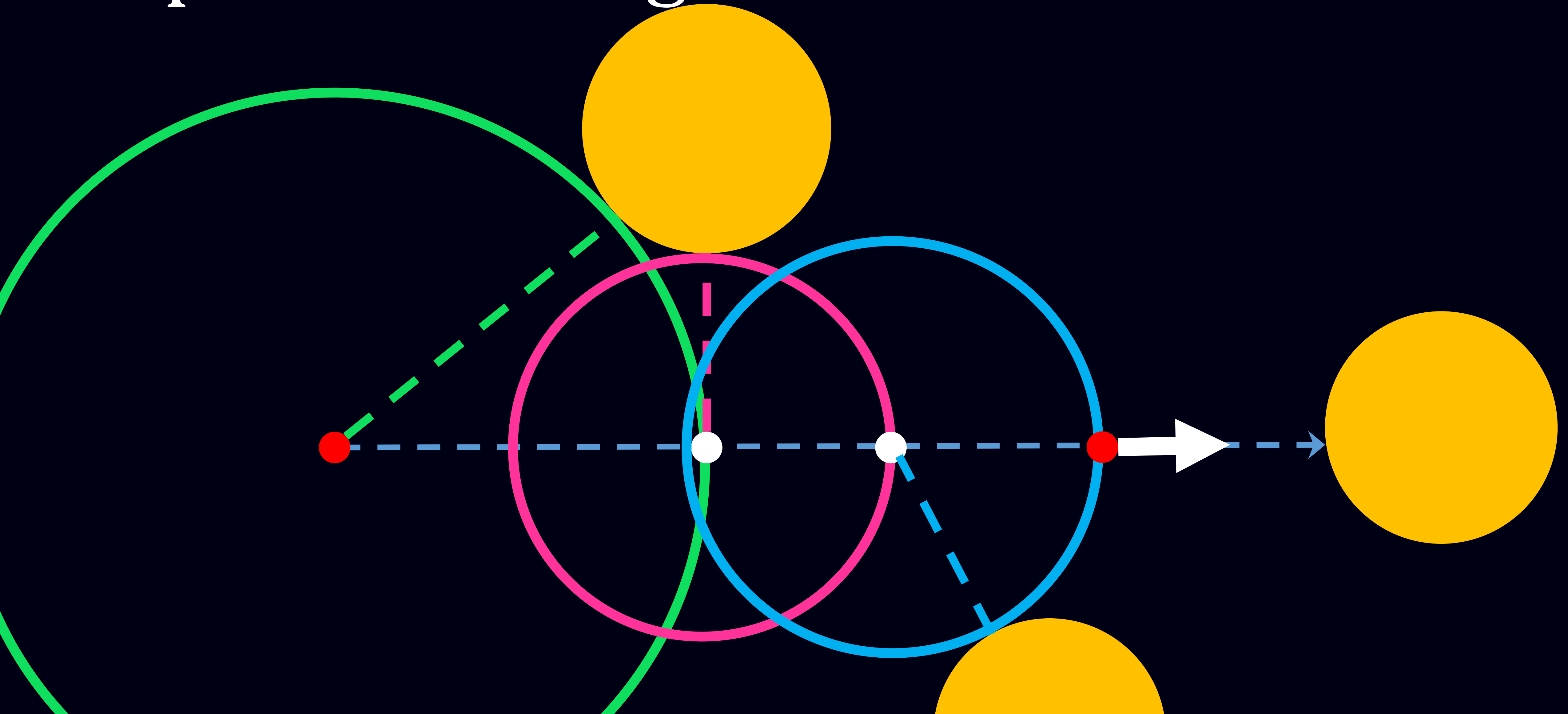
Sphere Tracing



Sphere Tracing



Sphere Tracing



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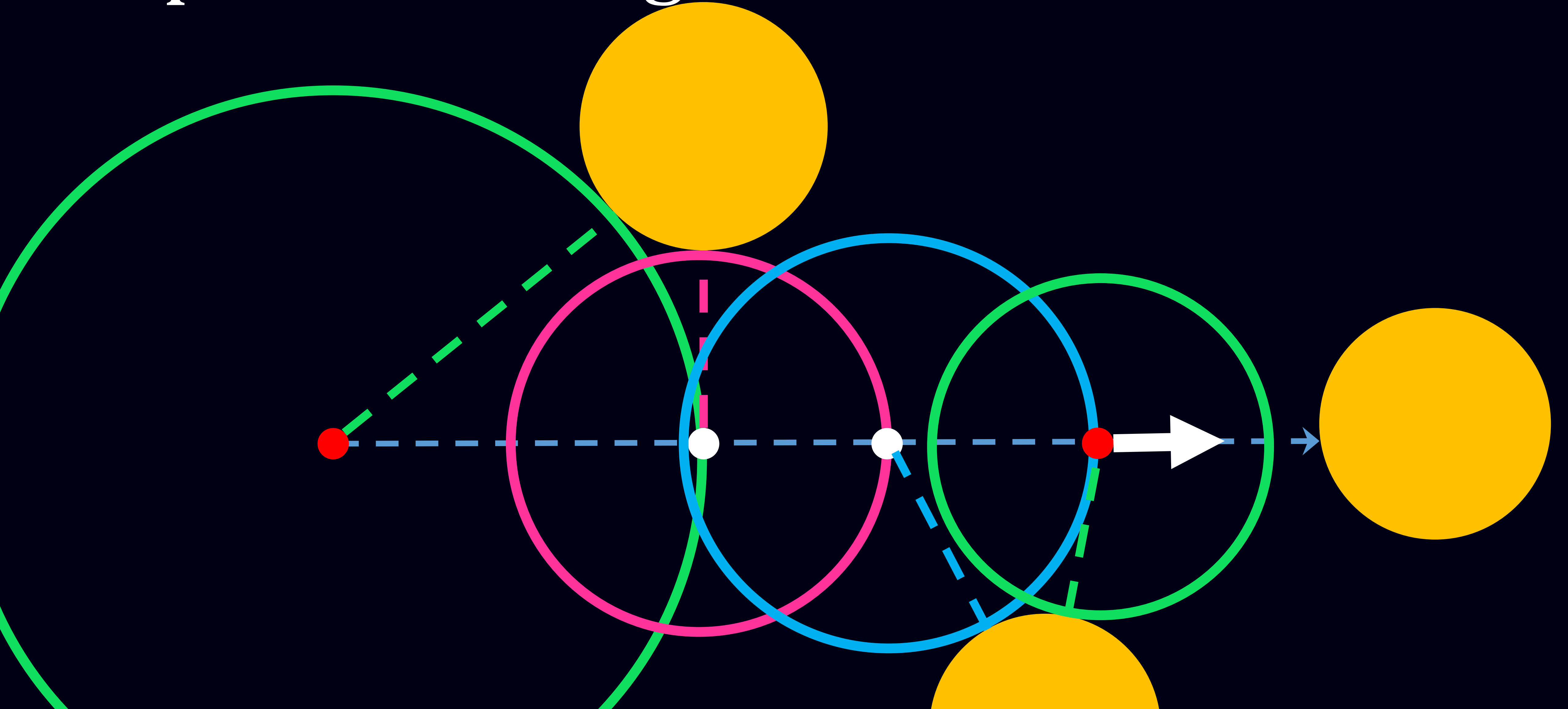
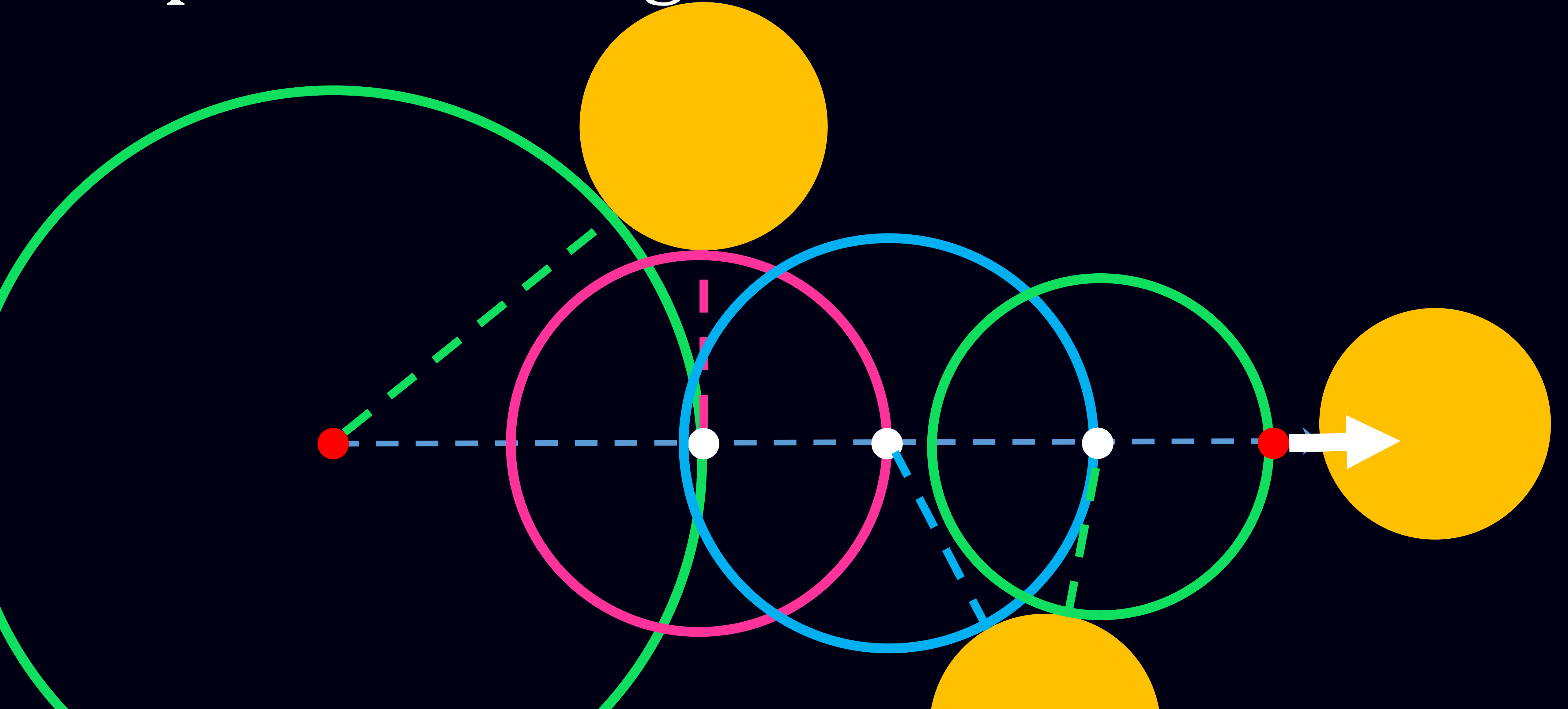


Sphere Tracing



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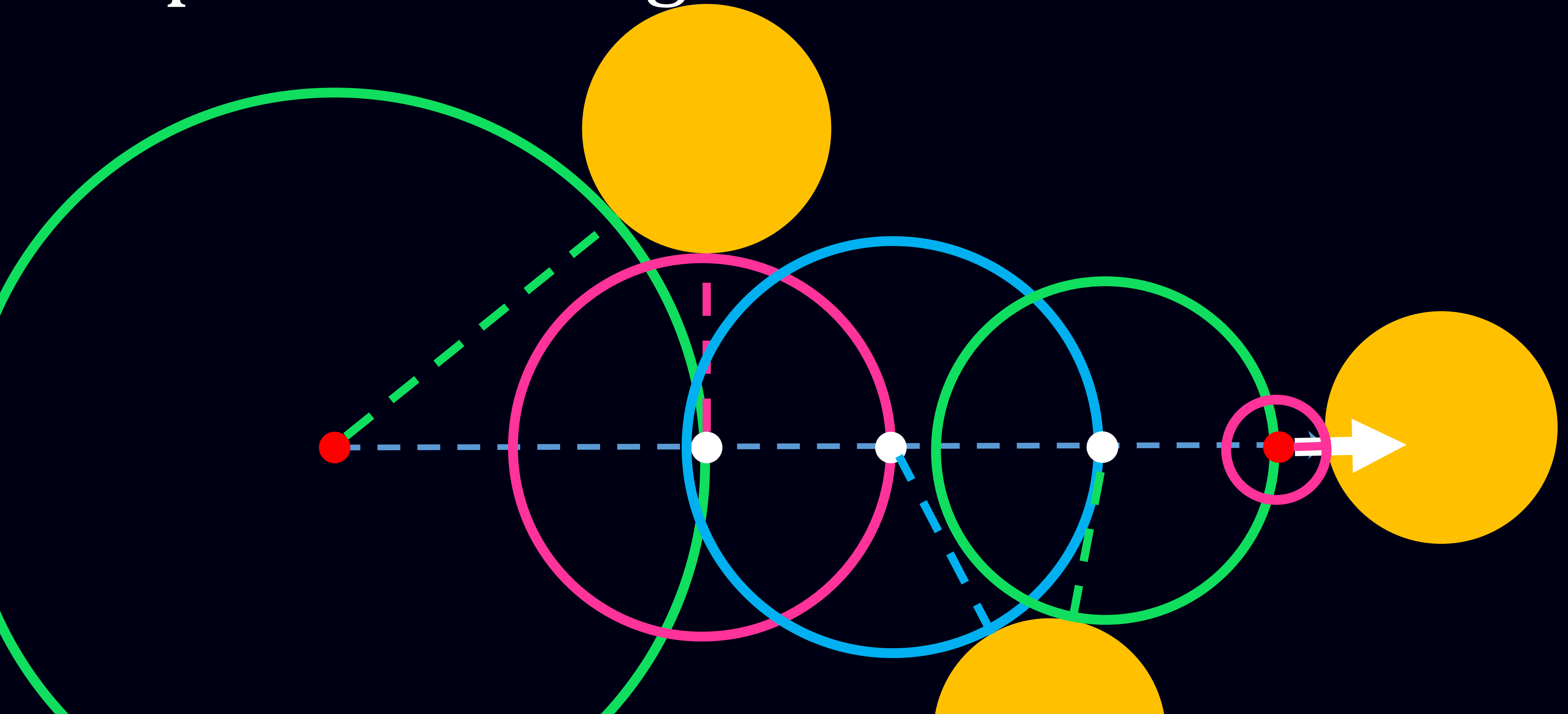
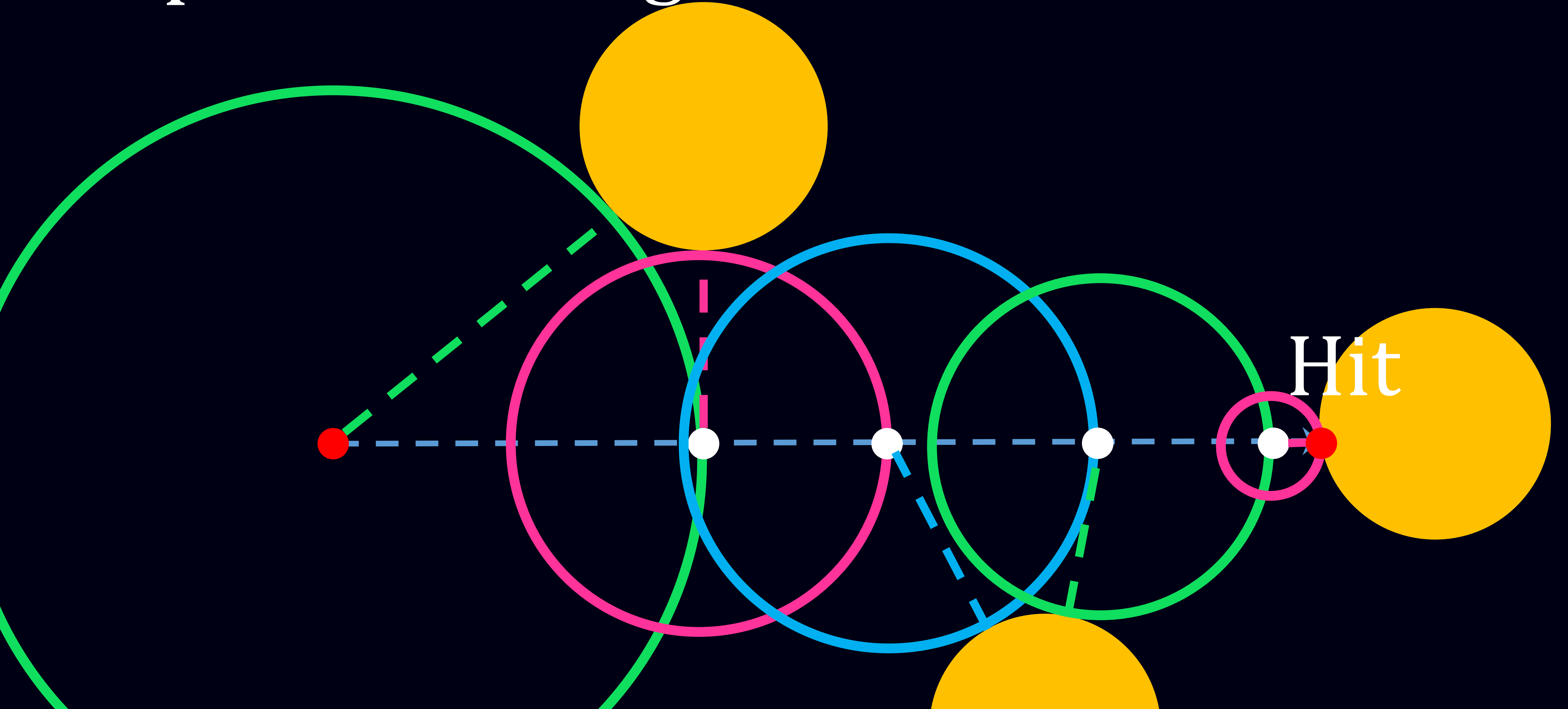


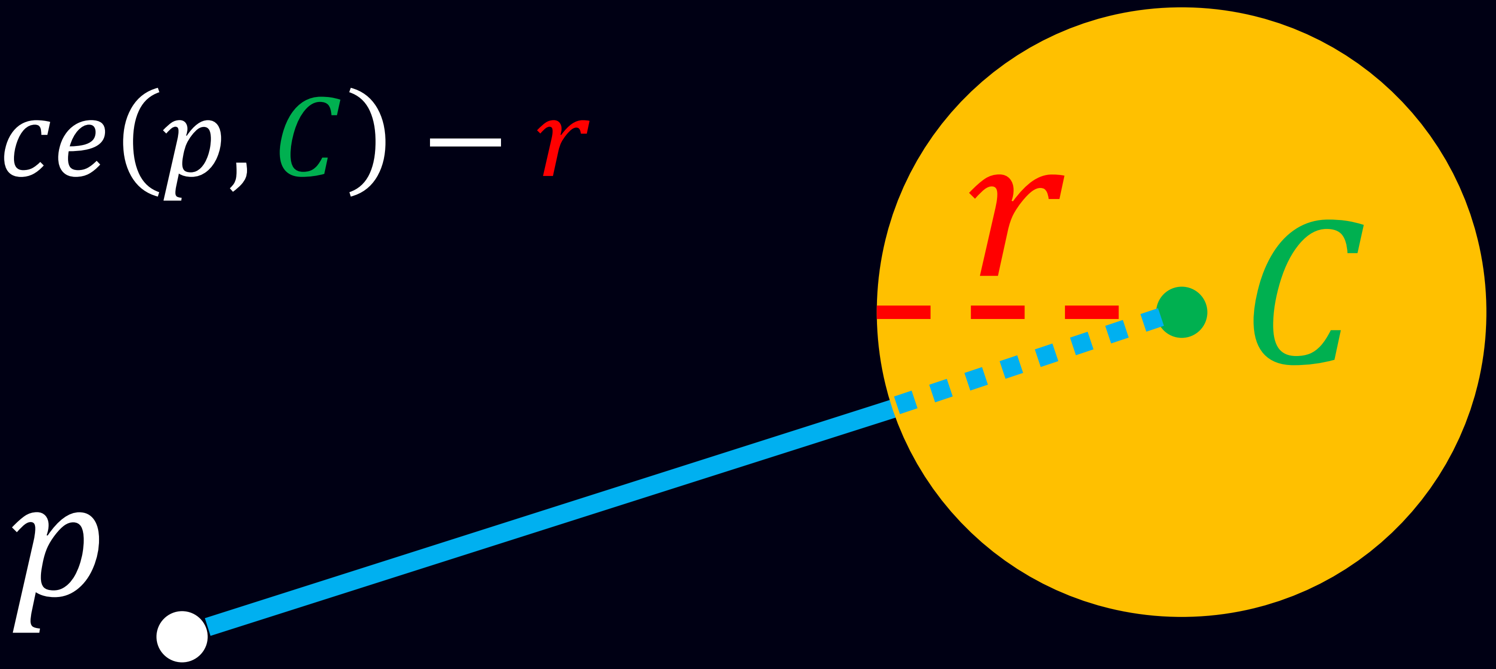
Sphere Tracing



Distance Function

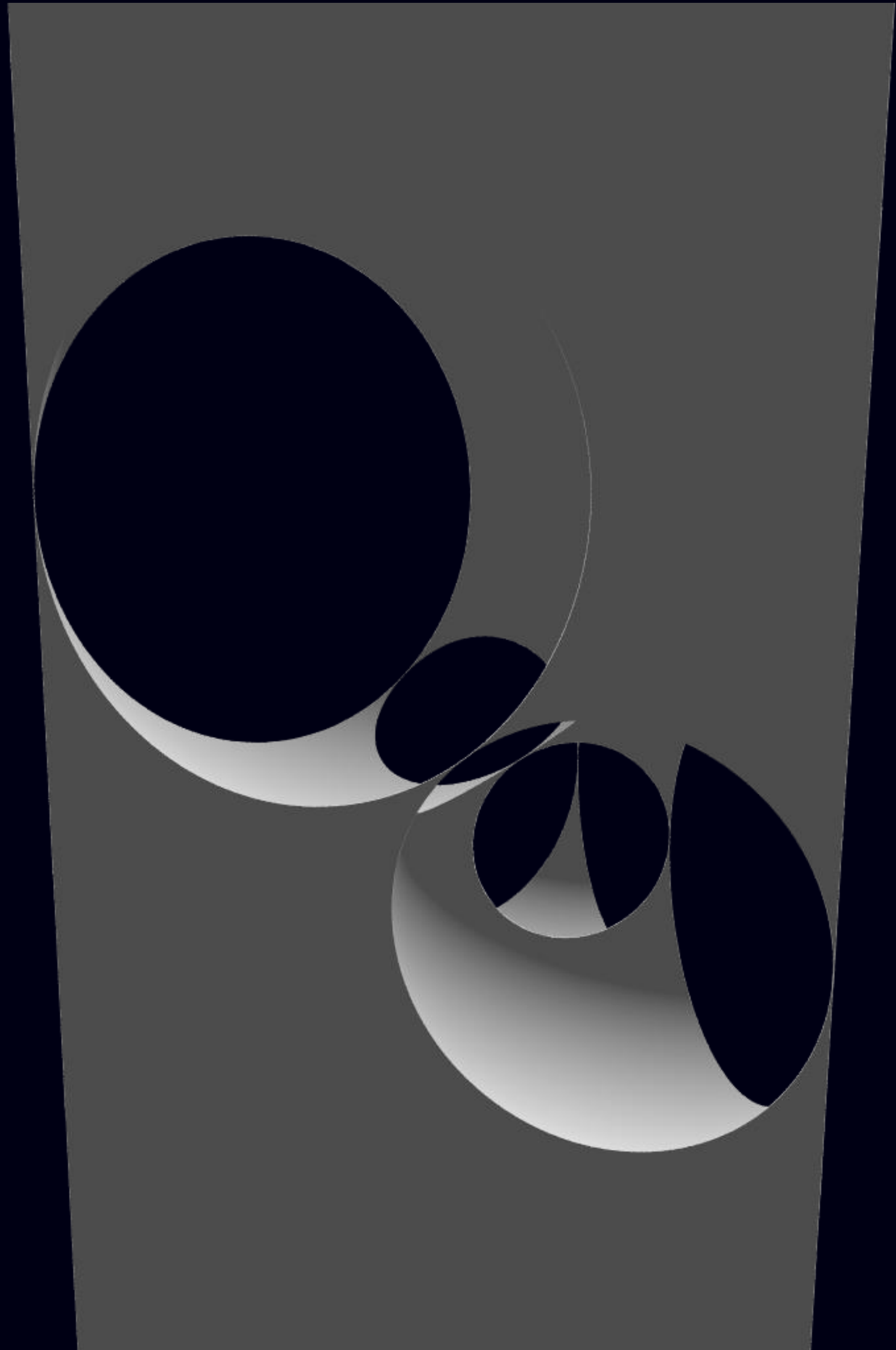
A function returning the minimum distance between given point and object's surface

$$f(p) = distance(p, C) - r$$



Distance to Sphairahedron

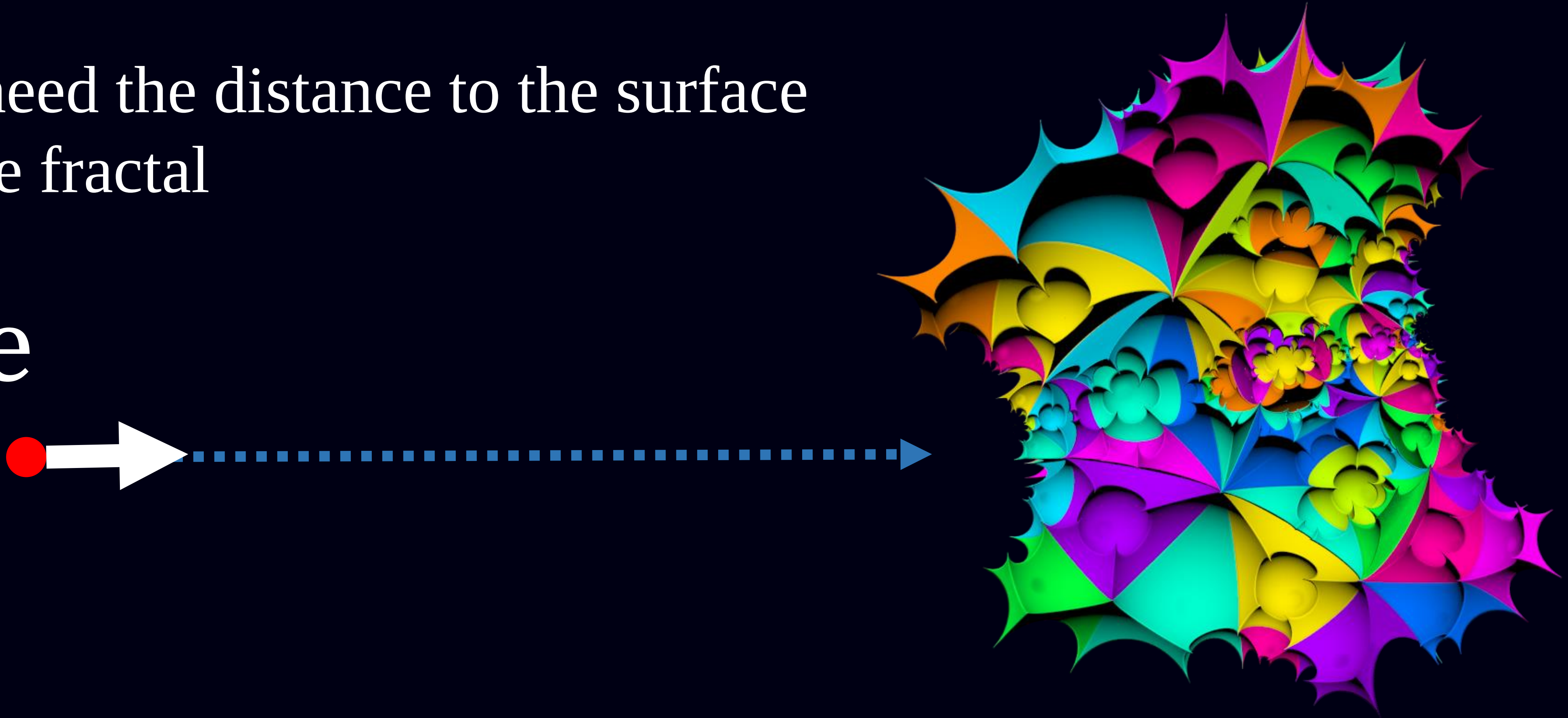
```
float DistanceToSphairahedron (vec3 p) {
        float d = DistanceToPrism(p);
        d = max(-DistanceToSphereA(p), d);
        d = max(-DistanceToSphereB(p), d);
        d = max(-DistanceToSphereC(p), d);
        return d;
}
```



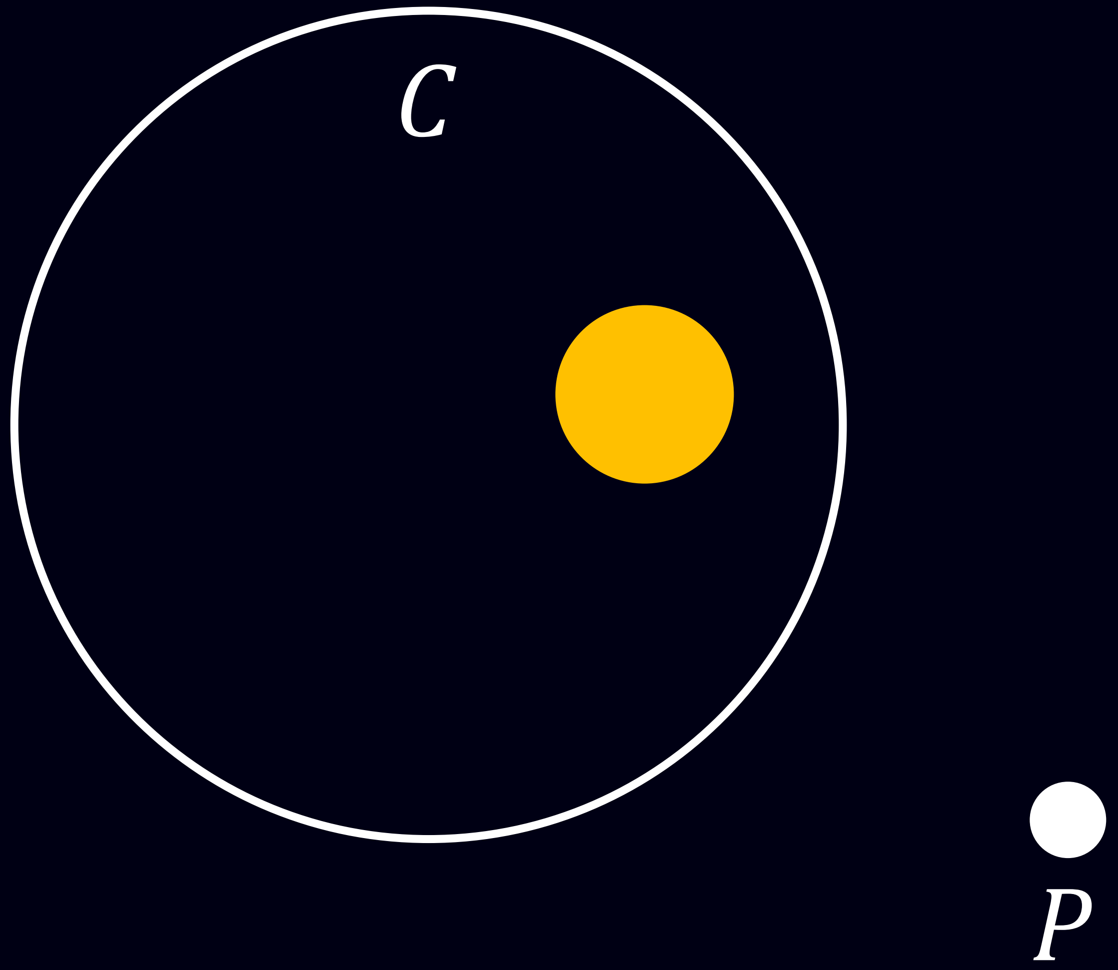
Ray Tracing

We need the distance to the surface of the fractal

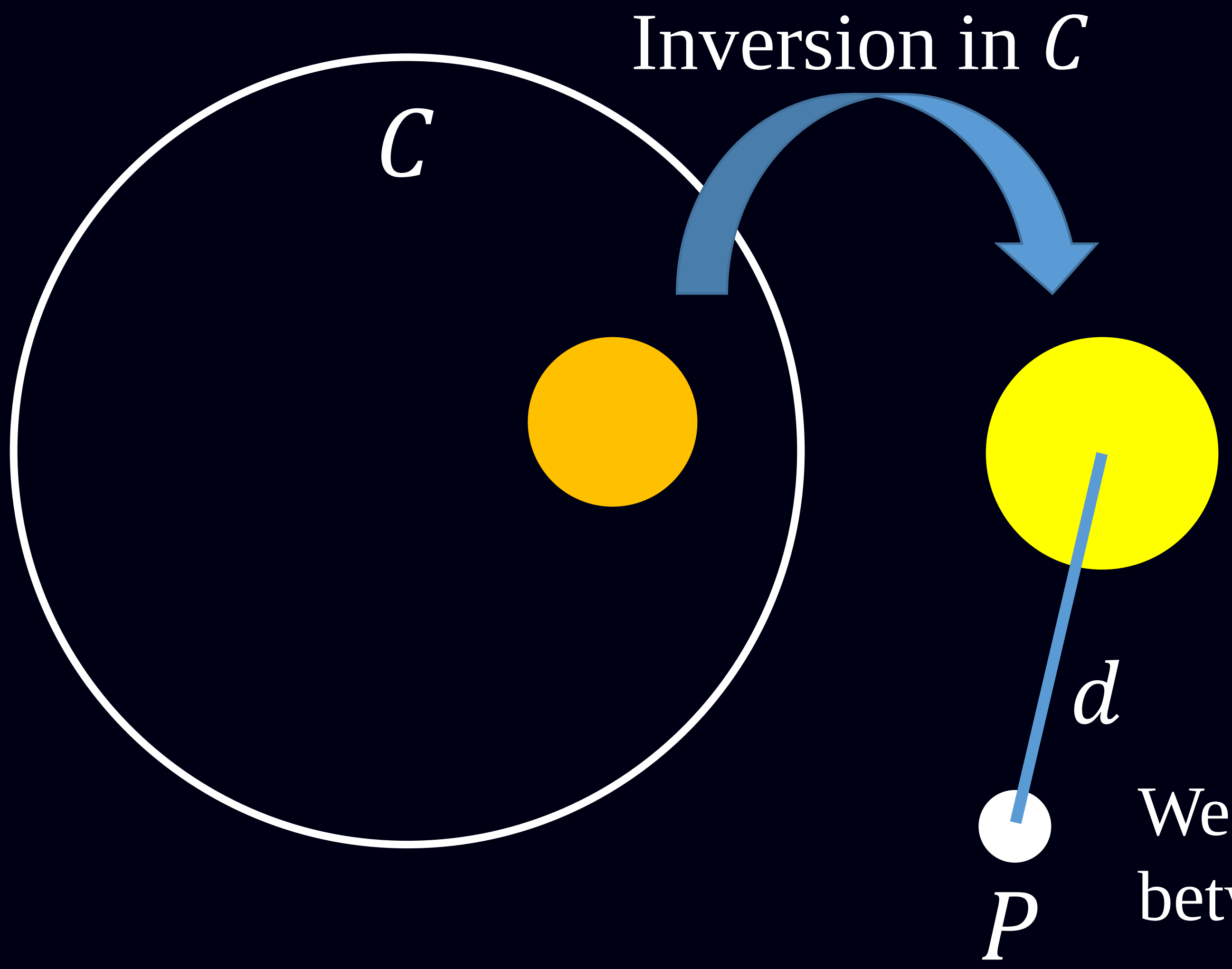
Eye



Distance Field for the orbit of spheres

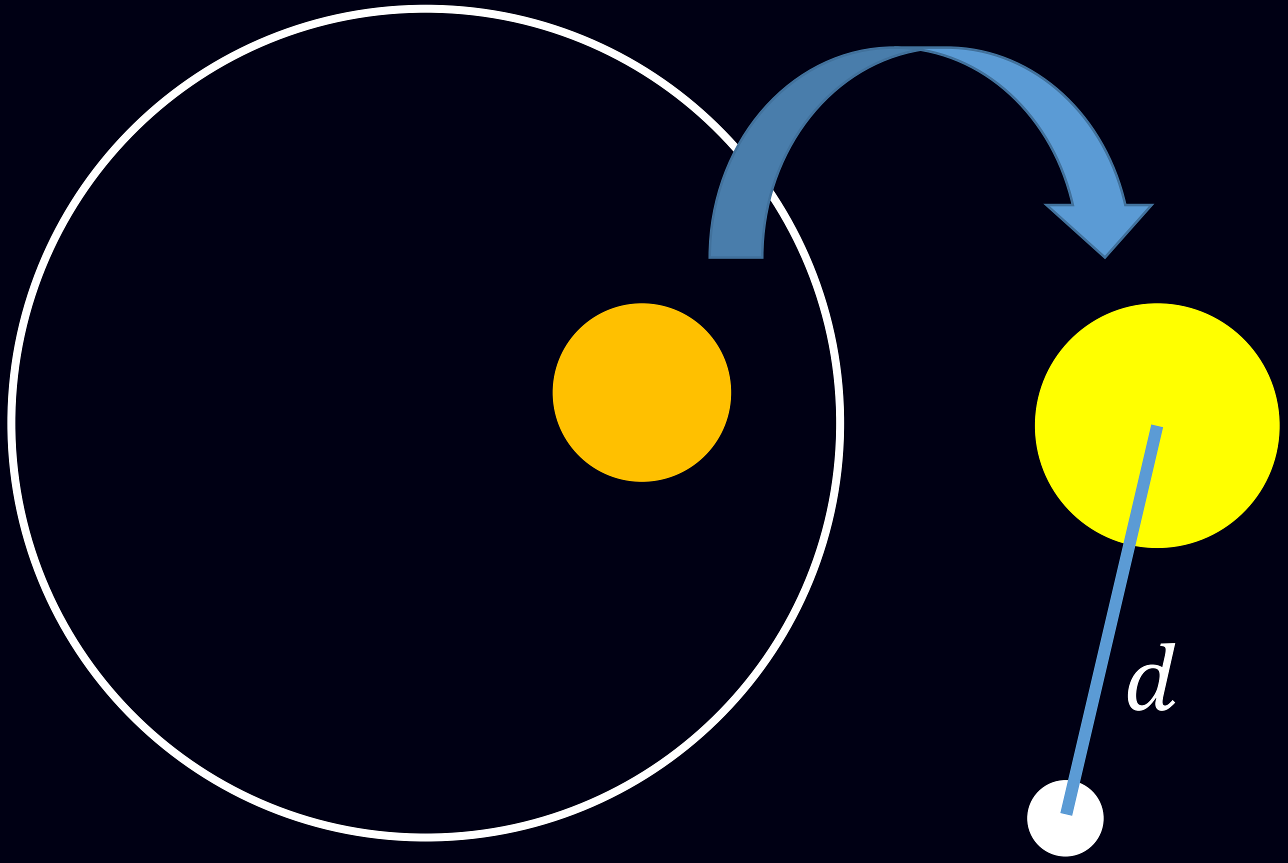


Distance Field for the orbit of spheres

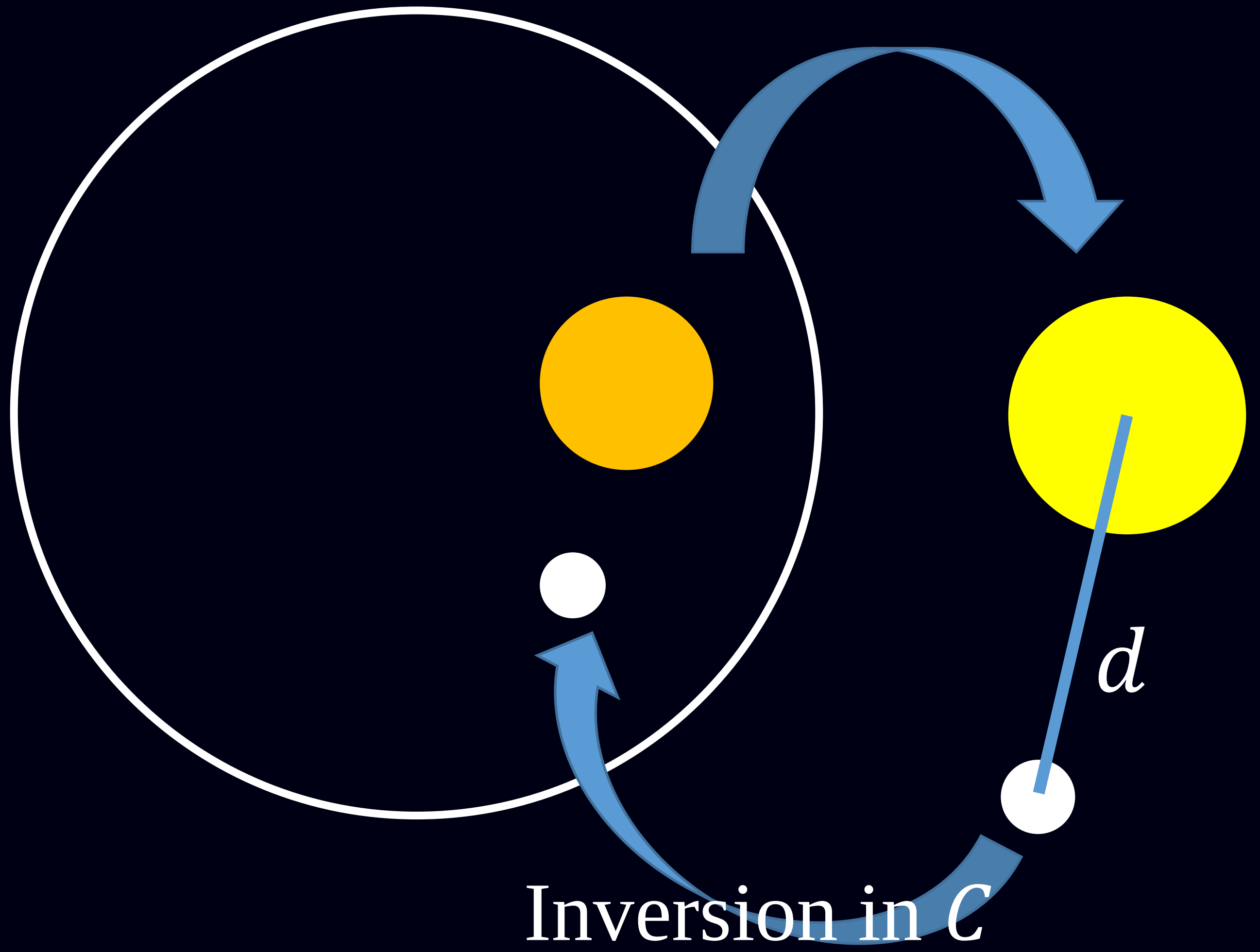


We need minimum distance between the point and spheres

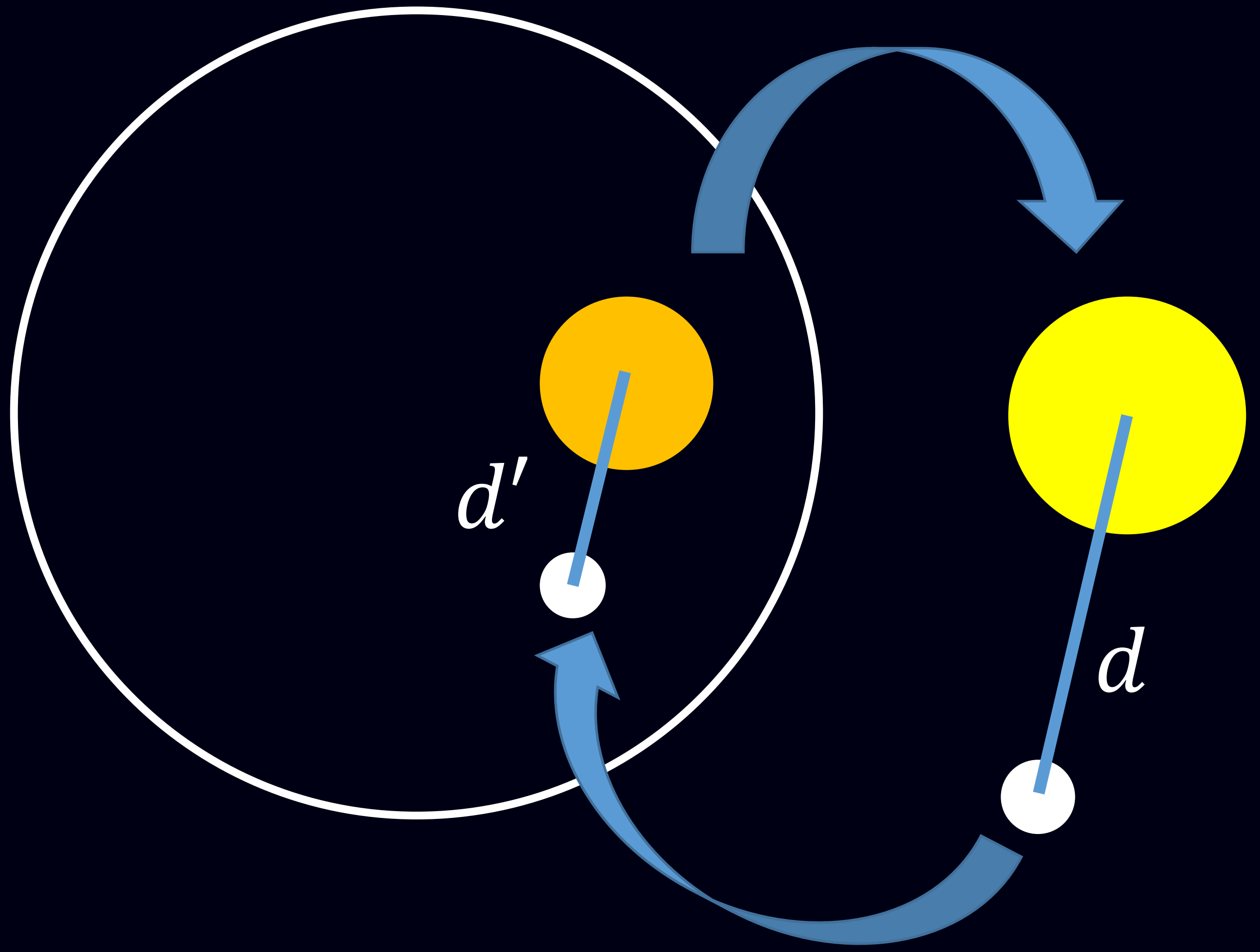
Distance Field for the orbit of spheres



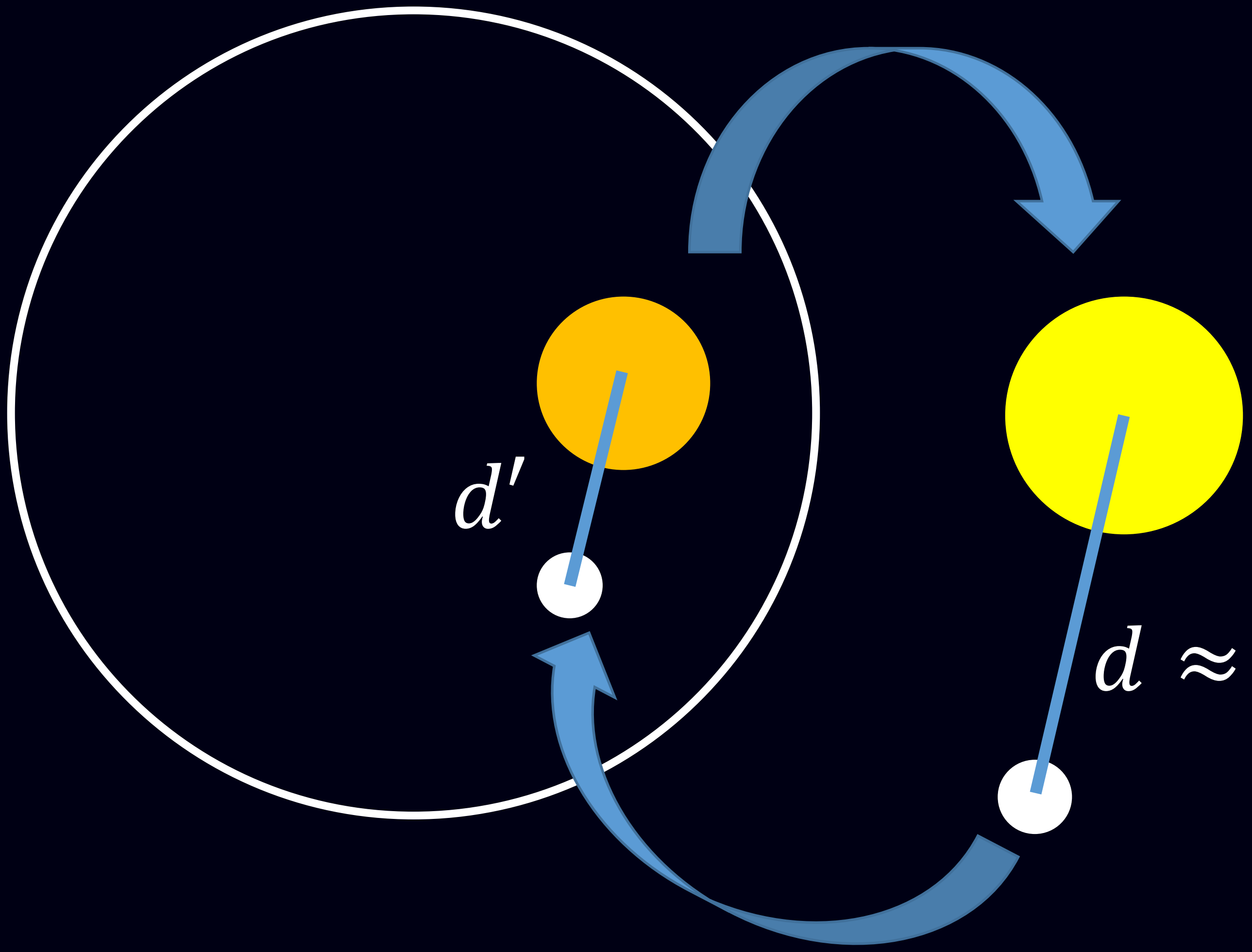
Distance Field for the orbit of spheres



Distance Field for the orbit of spheres



Distance Field for the orbit of spheres



$$d \approx \frac{d'}{Jacobian\ of\ InvC}$$

Experimental Sphairahedron Renderer

- https://soma-arc.net/SphairahedronExperiment/
- Environment … JavaScript + WebGL2.0
- Some parameters may require high GPU Power
- Source code

https://github.com/soma-arc/SphairahedronExperiment