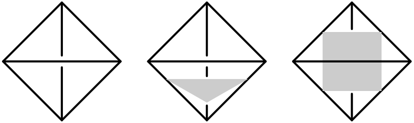
Telling 3-manifolds apart: new algorithms to compute Turaev-Viro invariants

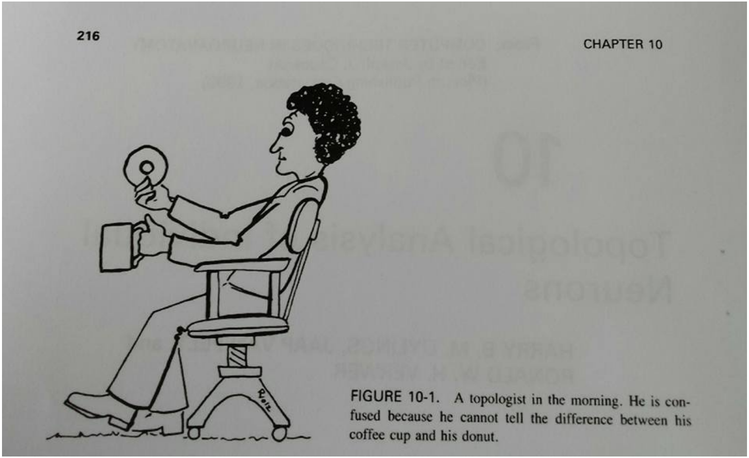
Jonathan Spreer (Freie Universität Berlin)

Topology and Computer, Osaka, October 21, 2017



Motivation

- This talk: study of manifolds up to (PL-)homeomorphism



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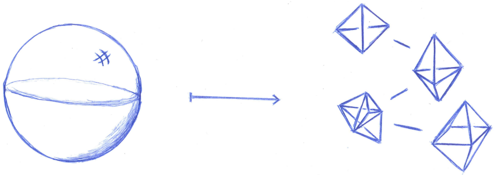
  manifolds $\leftrightarrow$ triangulated manifolds (simplicial?)



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- combinatorial arguments / discrete methods prove geometric and topological problems

- Fundamental task: distinguishing between manifolds, i.e., given triangulations M and N, is $M \ncong N$?

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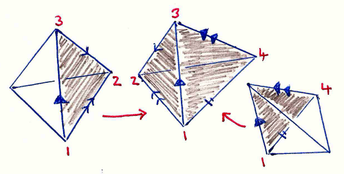

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Motivation

- Can we distinguish between manifolds?
  - Dimension 1: ✓
  - Dimension 2: ✓
  - Dimension 3: Yes in theory. No in general in practice.
  - Dimension ≥ 4: No.

- I.e., its trivial, extremely difficult, or impossible to distinguish between manifolds.

- Partial solution: topological invariants, properties of a manifold which do not change under continuous deformation

- Turaev-Viro invariants: particularly powerful family of topological invariants for 3-manifolds[1] [2]

- Method of choice when, for example, enumerating 3-manifolds

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[1] Matveev, Algorithmic Topology and Classification of 3-manifolds, 2003

[2] Kauffmann and Lins, Computing Turaev-Viro inv. for 3-manifolds, 1991

Turaev-Viro invariants

The Turaev-Viro invariant with parameters r and q is a function

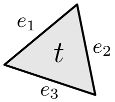
$$\mathrm{TV}_{r,q} : \mathcal{M} \to \mathbb{Q}[\zeta] \cap \mathbb{R}$$

where

- $\mathcal{M}$ = set of triangulated 3-manifolds (connected, closed)
- $\zeta = e^{i\pi q/r}$; $\quad r, q \in \mathbb{Z}$ co-prime; $\quad r \geq 3$; $\quad 0 < q < 2r$

- Can be computed via purely combinatorial formulae.

Turaev-Viro invariants – state-sum model

- $M \in \mathcal{M}$ triangulated 3-manifold
- V, E, F, T its set of vertices, edges, triangles, and tetrahedra
- $\varphi : E \to \{0, 1, \ldots, r-2\}$ edge colouring satisfying the following conditions at all triangles t of M:
- 

$$\begin{array}{lcll} \varphi(e_i) + \varphi(e_j) & \geq & \varphi(e_k) & \forall i \neq j \neq k \neq i \\ \sum \varphi(e_i) & \equiv & 0 \mod 2 & \text{and } \leq 2r-4 \end{array}$$

- Call the set of such admissible colourings $\mathrm{Adm}(M, r)$
- For each $\varphi \in \mathrm{Adm}(M, r)$, edge $e \in E$, triangle $t \in F$, and tetrahedron $\Delta \in T$ we define weights $|e|_\varphi$, $|t|_\varphi$, and $|\Delta|_\varphi$ in $\mathbb{Q}[\zeta]$ only depending on φ (and r and q)
- $$\mathrm{TV}_{r,q}(M) = \sum_{\varphi \in \mathrm{Adm}(M,r)} \left(\prod_{e \in E} |e|_\varphi \cdot \prod_{t \in F} |t|_\varphi \cdot \prod_{\Delta \in T} |\Delta|_\varphi \right)$$

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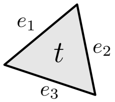
- 

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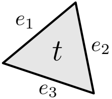
- 

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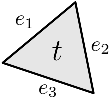
- 

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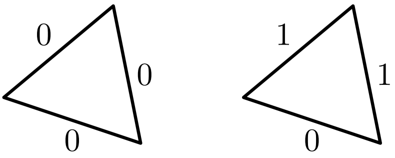
An alternative view on admissible colourings

$r = 3$ (colours 0, 1) $\in$ **P**:



$r = 4$ (colours 0, 1, 2) $\in$ **#P-hard**[3]:

[3]Kirby, Melvin, Local surgery formulas for quantum invariants and the Arf invariant, 2004.

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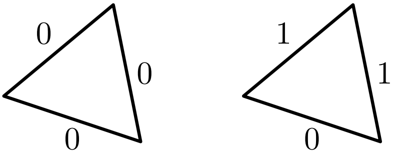

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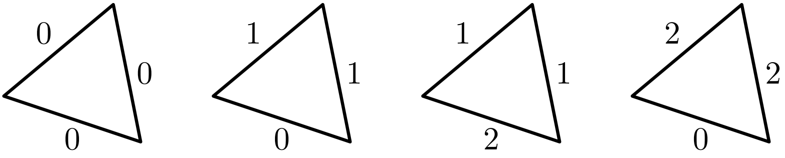

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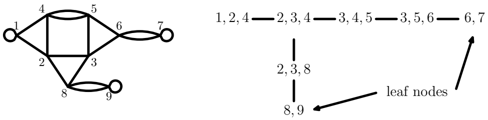
Algorithm I: treewidth

- The treewidth of a graph measures how "treelike" a graph is (trees have treewidth 1)
- The treewidth of a triangulated manifold M is the treewidth of its dual graph
- Low treewidth $\Rightarrow$ can arrange tetrahedra of M in a tree with few tetrahedra grouped together per node of the tree ($\Rightarrow$ thin tree decomposition)



- Suitable for dynamic programming

Algorithm I: treewidth

Idea:

- Given a triangulation, compute a tree decomposition with few tetrahedra per node (if possible)
- Enumerate admissible colourings and weights from the leave nodes up
- Grouping partial colourings together wherever they look the same at the current node

Algorithm I: treewidth

Theorem (Burton, Maria, S. 2015)

Given a triangulated 3-manifold M with n tetrahedra, and a tree decomposition of M with largest node of size k, we can compute $\mathrm{TV}_{r,q}$ *in*

$$O\left(n \cdot (r-1)^{6k} \cdot k^2 \cdot \log r\right).$$

- Running time is of type $g(k) \times \mathrm{poly}(n)$. In the literature such an algorithm is referred to as fixed parameter tractable (FPT)[4] in k ("treewidth")
- Common for FPT algorithms is a very bad parameter function $g : \mathbb{N} \to \mathbb{N}$ (tower of exponentials)
- Here: $g(k) = (r-1)^{6k} \cdot k^2 \cdot \log r$ vs. $(r-1)^{|E|}$
- This is why we implemented the algorithm (also very rare for FPT algorithms)

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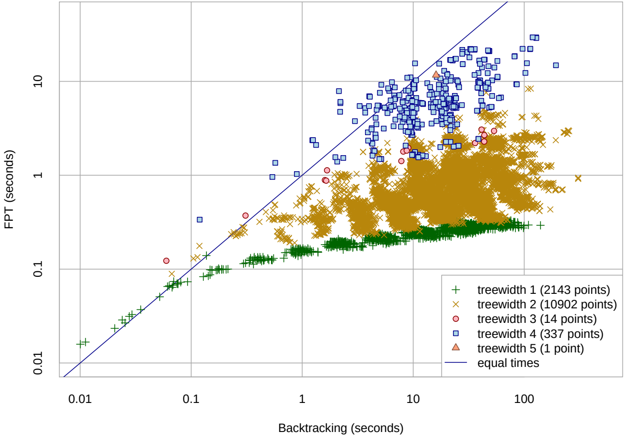
[4] Downey, Fellows, Parameterized complexity, Springer

Algorithm I: treewidth



Running times for $TV_{7,1}$ for the minimal 11-tetrahedra triangulations of closed prime orientable 3-manifolds.

Observations

GOOD:

- works for all parameters r and q
- faster than naive enumeration

NOT GOOD:

- properties of triangulation, not manifold, determine running time: "*every manifold admits a triangulation with arbitrarily high treewidth*"
- exact treewidth might be difficult to determine
- algorithm requires large amounts of memory

BETTER:

- Use parameter which is also a topological invariant
- Easy to compute, even if large

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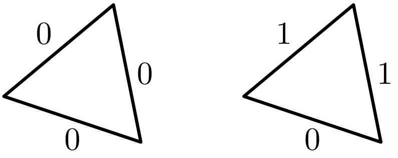

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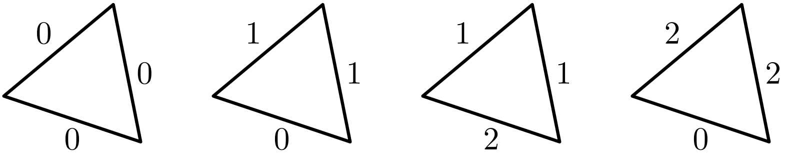

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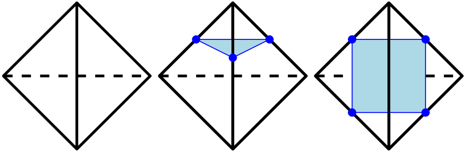

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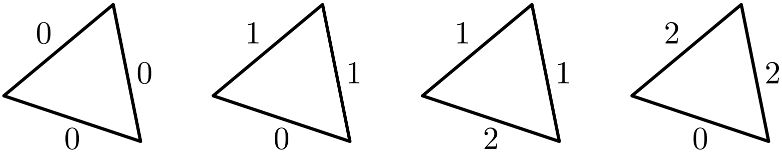

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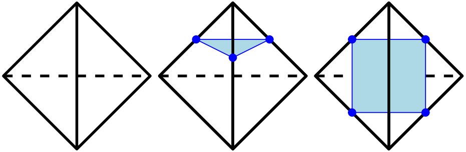

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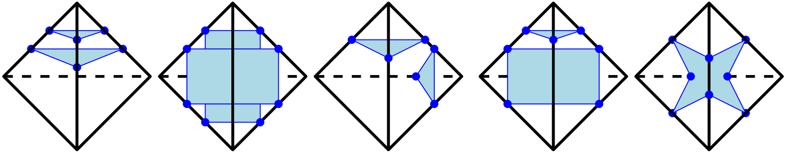

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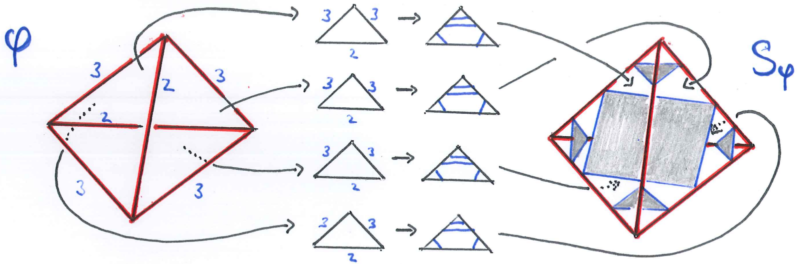

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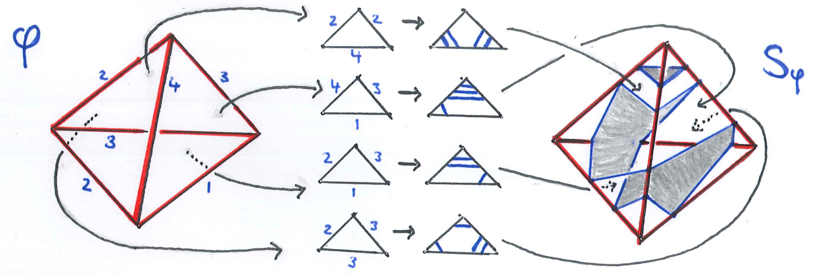

Algorithm II: $\beta_1(M, \mathbb{Z}_2)$

Lemma (Maria, S. 2016)

Let $\varphi \in \mathrm{Adm}(M, 4)$ *and let* φ_0 *be the reduction of* φ *(i.e., all colors mod 2). Then*

$$|M|_\varphi = (-1)^\alpha (\pm\sqrt{2})^{\chi(S_{\varphi_0})},$$

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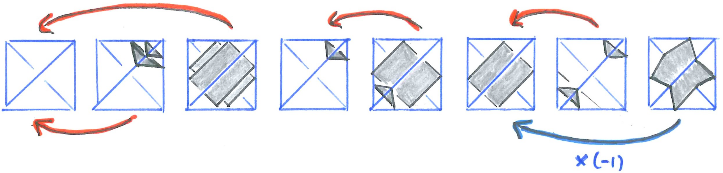

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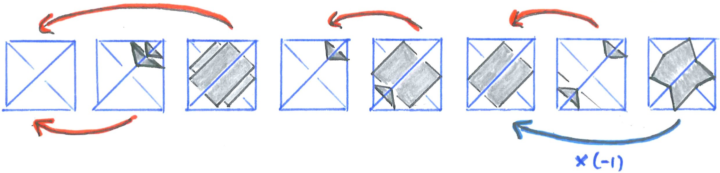

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Theorem (Maria, S. 2017)

M 1-vertex, n-tetrahedra triangulated 3-manifold with first Betti number $\beta_1(M, \mathbb{Z}_2)$. Then there exists an algorithm to compute $\mathrm{TV}_{4,q}(M)$ with running time

$$O(2^{\beta_1(M,\mathbb{Z}_2)} n^3)$$

in $O(n^2)$ memory and with $O(2^{\beta_1(M,\mathbb{Z}_2)})$ cyclotomic field operations.

Practical improvements:

	New algo.	Treewidth algo.[6]	$\mathbb{Z}$-hom. in Regina
≤ 11 tet. census	10.96 sec.	498 sec.	7.72 sec.

Theoretical improvements: Distinguishes roughly twice as many manifolds as $\mathbb{Z}$-homology on its own

[6] Burton, Maria, S., Algorithms and complexity for Turaev-Viro inv's., 2015

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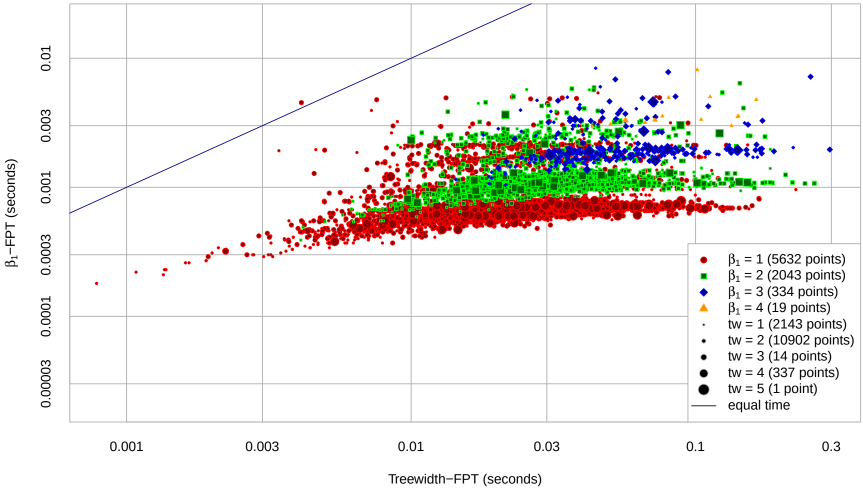

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Thank you

Benjamin A. Burton, Clément Maria, Jonathan Spreer, *Algorithms and complexity for Turaev-Viro invariants*. Automata, Languages, and Programming: 42nd International Colloquium, ICALP 2015, Kyoto. Proceedings, Part 1, pg. 281–293. arXiv:1503.04099.

Clément Maria, Jonathan Spreer, *A polynomial time algorithm to compute quantum invariants of* 3*-manifolds with bounded first Betti number*. Proceedings of the ACM-SIAM Symposium on Discrete Algorithms (SODA 2017), pg. 2721–2732. arXiv:1607.02218.