Cubical Ripser

- A calculator of the persistent homology of the cubical complex

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Motivation

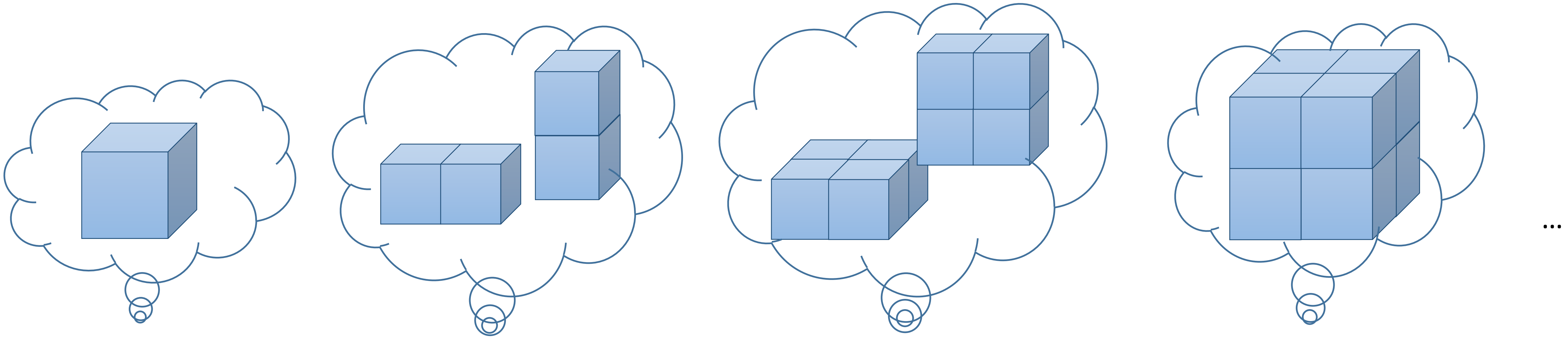
- CREST ソフトマター記述言語の創造に向けた位相的データ解析理論の構築
- DIPHA, PHAT, Perseus : calculators of the persistent homology
  - It takes time to calculate the persistent homology of huge amount of data.

So we need a new software to calculate more quickly.

Cubical complex

Simplicial complex is a set consisting of points, line segments and <u>triangles</u>.

Cubical complex is a set composed of squares or cubes.



0-cell 1-cell 2-cell 3-cell …

Example of cubical complex

3	2	2	3	1	-1	-1	-1	1
2	2	0	3	3	1	1	1	1
2	3	0	-1	4	3	4	2	2
2	0	0	-1	-1	-1	0	-2	-2
2	1	-1	-2	1	0	0	-2	-3
1	1	-2	-2	1	-2	-2	-3	-3
1	1	-2	-2	-2	-2	-3	-3	-3

Threshold: -3

3	2	2	3	1	-1	-1	-1	1
2	2	0	3	3	1	1	1	1
2	3	0	-1	4	3	4	2	2
2	0	0	-1	-1	-1	0	-2	-2
2	1	-1	-2	1	0	0	-2	-3
1	1	-2	-2	1	-2	-2	-3	-3
1	1	-2	-2	-2	-2	-3	-3	-3

Dim 0: [-3,

Threshold: -2

3	2	2	3	1	-1	-1	-1	1
2	2	0	3	3	1	1	1	1
2	3	0	-1	4	3	4	2	2
2	0	0	-1	-1	-1	0	-2	-2
2	1	-1	-2	1	0	0	-2	-3
1	1	-2	-2	1	-2	-2	-3	-3
1	1	-2	-2	-2	-2	-3	-3	-3

Dim 0: [-3,

Threshold: -1

3	2	2	3	1	-1	-1	-1	1
2	2	0	3	3	1	1	1	1
2	3	0	-1	4	3	4	2	2
2	0	0	-1	-1	-1	0	-2	-2
2	1	-1	-2	1	0	0	-2	-3
1	1	-2	-2	1	-2	-2	-3	-3
1	1	-2	-2	-2	-2	-3	-3	-3

Dim 0: [-3,), [-1,

Threshold: 0

3	2	2	3	1	-1	-1	-1	1
2	2	0	3	3	1	1	1	1
2	3	0	-1	4	3	4	2	2
2	0	0	-1	-1	-1	0	-2	-2
2	1	-1	-2	1	0	0	-2	-3
1	1	-2	-2	1	-2	-2	-3	-3
1	1	-2	-2	-2	-2	-3	-3	-3

Dim 0: [-3,), [-1,

Dim 1: [0,

Threshold: 1

3	2	2	3	1	-1	-1	-1	1
2	2	0	3	3	1	1	1	1
2	3	0	-1	4	3	4	2	2
2	0	0	-1	-1	-1	0	-2	-2
2	1	-1	-2	1	0	0	-2	-3
1	1	-2	-2	1	-2	-2	-3	-3
1	1	-2	-2	-2	-2	-3	-3	-3

Dim 0: [-3,), [-1,

Dim 1: [0, 1)

Threshold: 2

3	2	2	3	1	-1	-1	-1	1
2	2	0	3	3	1	1	1	1
2	3	0	-1	4	3	4	2	2
2	0	0	-1	-1	-1	0	-2	-2
2	1	-1	-2	1	0	0	-2	-3
1	1	-2	-2	1	-2	-2	-3	-3
1	1	-2	-2	-2	-2	-3	-3	-3

Dim 0: [-3,), [-1, 2)

Dim 1: [0, 1), [2,

Threshold: 3

3	2	2	3	1	-1	-1	-1	1
2	2	0	3	3	1	1	1	1
2	3	0	-1	4	3	4	2	2
2	0	0	-1	-1	-1	0	-2	-2
2	1	-1	-2	1	0	0	-2	-3
1	1	-2	-2	1	-2	-2	-3	-3
1	1	-2	-2	-2	-2	-3	-3	-3

Dim 0: [-3,), [-1, 2)

Dim 1: [0, 1), [2, 3), [3,), [3,)

Threshold: 4

3	2	2	3	1	-1	-1	-1	1
2	2	0	3	3	1	1	1	1
2	3	0	-1	4	3	4	2	2
2	0	0	-1	-1	-1	0	-2	-2
2	1	-1	-2	1	0	0	-2	-3
1	1	-2	-2	1	-2	-2	-3	-3
1	1	-2	-2	-2	-2	-3	-3	-3

Dim 0: [-3, ∞), [-1, 2)

Dim 1: [0, 1), [2, 3), [3, 4), [3, 4)

Ripser

- A software for computing the persistence barcodes in Vietoris-Rips complex
- Made by Ulrich Bauer in 2016
- The main feature
  - about 1000 lines of code in C++, no external dependencies
  - support for coefficients in prime fields
  - time- and memory-efficient

- Open source (http://ripser.org)

Cubical Ripser

- A software for computing persistence pairs in cubical complex (C++)
  - ➢For both of two-dimensional and three-dimensional data, that is, gray-scale pixel(or voxel) data.

- Using Ripser’s algorithm
  - ➢coface, pivot, compute pairs, assemble columns to reduce etc
  - ➢Vietoris-Rips complex → cubical complex

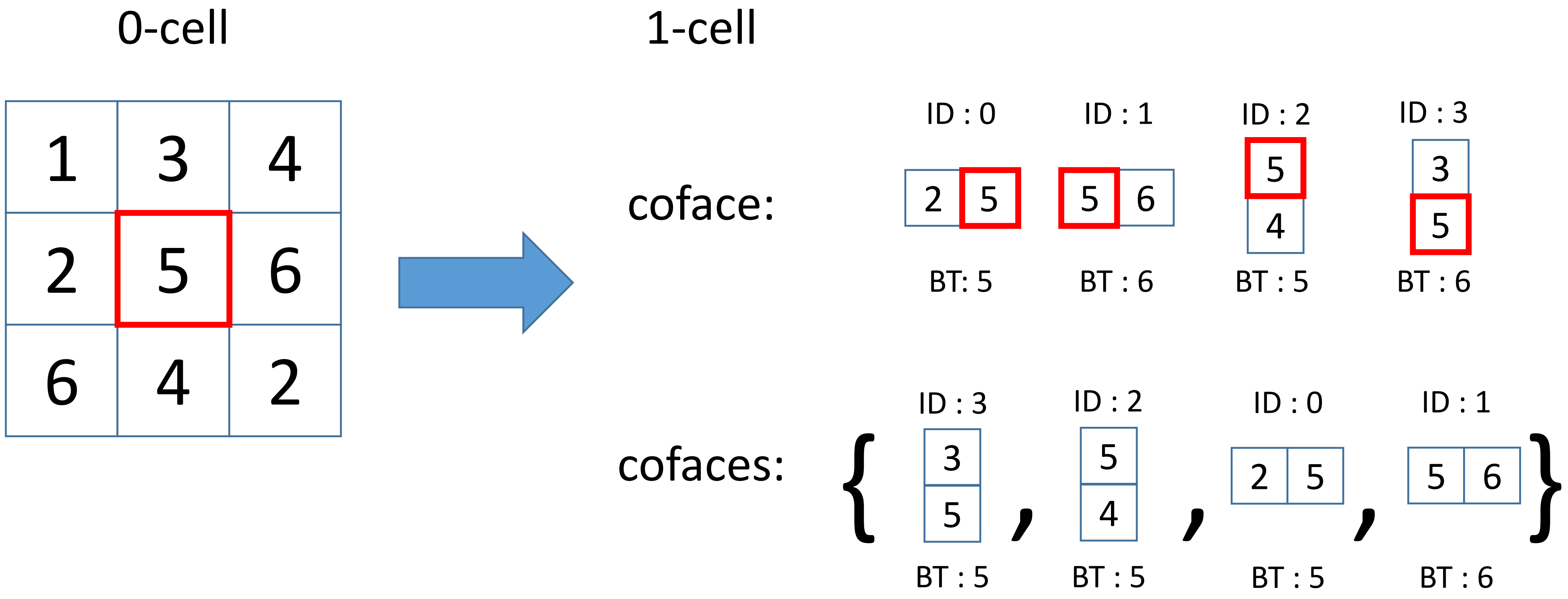
Coface

- Each cell has an index(ID) and a birth time(BT).
  - Ex)   1-cell

$$birth\ time = 5$$

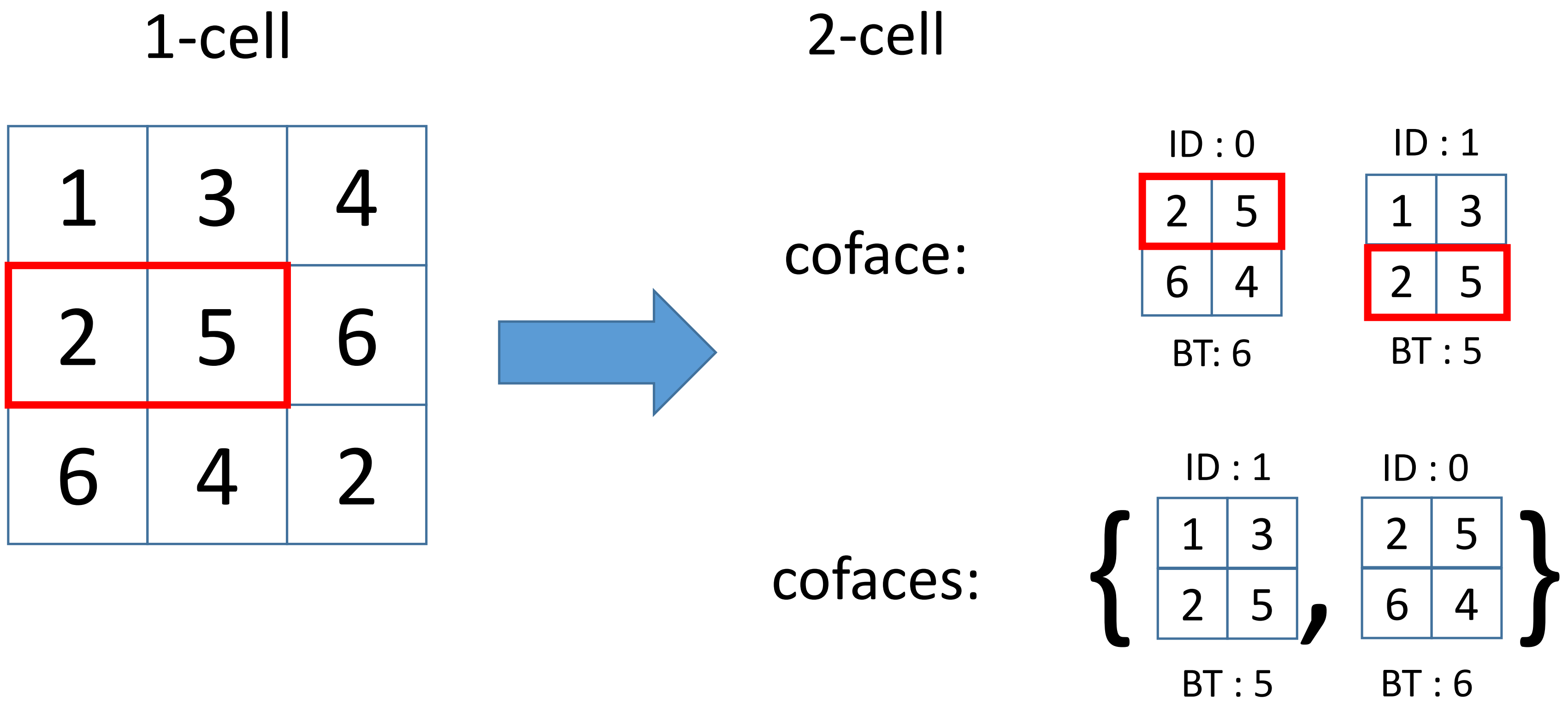
- For each d-cell σ, coface is defined by $(d+1)$-cell which includes σ and 'cofaces' is a list of the cofaces sorted in dictionary order of "smaller BT or greater ID".

Example of coface



Note : The BT of a coface is equal to or larger than that of the original.

Example of coface



Note : The BT of a coface is equal to or larger than that of the original.

Pivot (conventional)

For a boundary matrix $M \in \mathbb{Z}_2^{n \times m}$, we let M_j denote its j-th column, and $M_j^i \in \mathbb{Z}_2$ its entry in row i and column j. We set

$$pivot(M_j) := \max(i = 1, \dots, n | m_i = 1)$$

and call it the pivot index of that column.

- Homology ($\mathrm{H}_d = \mathrm{Z}_d / \mathrm{B}_d$)

$$\begin{array}{cc} & C_{d+1} \\ \mathrm{Z}_d & \begin{pmatrix} & 1 & 1 & & \\ 1 & & & 1 & 1 \\ & & & \textcircled{1} & 1 \\ \textcircled{1} & & 1 & & \\ & \textcircled{1} & & & \end{pmatrix} \end{array}$$

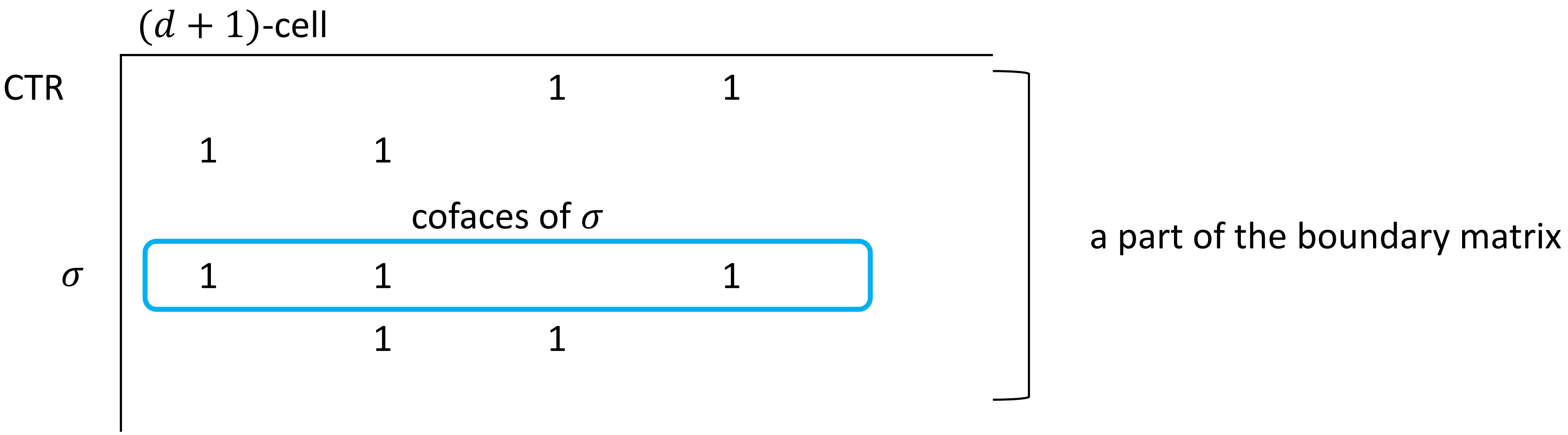
Pivot $\leftrightarrow$ generator of B_d

Pivot and Compute pairs(CP)

- Make <u>columns to reduce(CTR)</u>

  ↪ a list of d-cells (sorted in ascending order of BT)

  ↔ a list of generators of Z_d



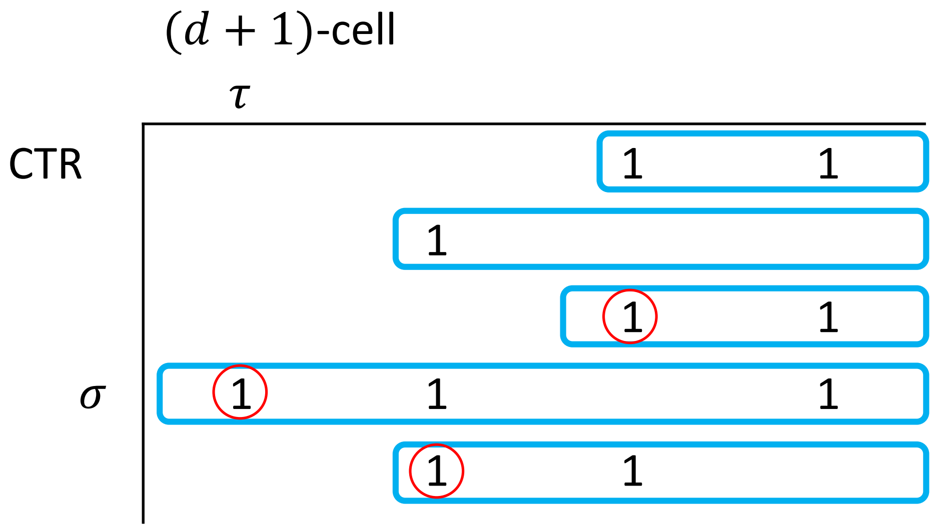
Columns are the list of all $(d+1)$-cells in dictionary order of (BT, ID).

Pivot and Compute pairs

- Look at boundary matrix from the bottom row to the top row.

Ⅰ. If the first (leftmost) coface τ of σ isn't contained in the list of pivots, then add τ to the list of pivots.

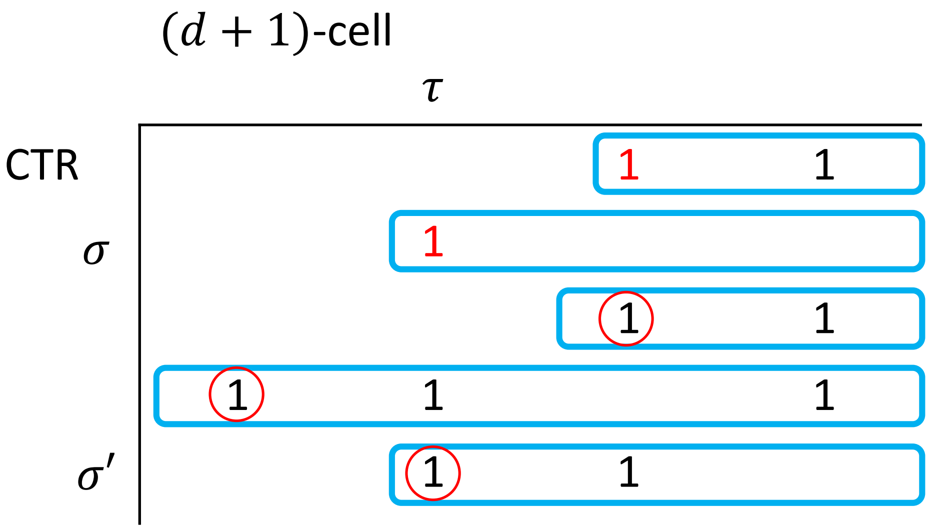
$If\ (BT\ of\ \sigma) < (BT\ of\ \tau) \rightarrow [BT\ of\ \sigma,\ BT\ of\ \tau)$ is added to the list of persistence pairs(PP)



Red circles mean that the element is on the position of pivots.

Pivot and Compute pairs

Ⅱ. If the first coface τ of σ has been already included in the list of pivots, add the row σ' to the row σ and look for the first coface again



τ is the first coface and τ is the pivot of the row σ'.

Example of pivot and Compute pairs

	30 τ_1		40 τ_3	40 τ_2	80 τ_4	90 τ_5	
σ_1(BT: 30)	(1)	1					
σ_2(BT: 34)			1			1	
σ_3(BT: 40)			(1)	1			
σ_4(BT: 80)					(1)	1	
σ_5(BT: 90)						(1)	1

$\mathbb{Z}_2$-coefficient

+	0	1
0	0	1
1	1	0

The coface of σ_2 has already been pivot (the coface of σ_3)

$$\sigma_2 \leftarrow \sigma_2 + \sigma_3$$

	τ_3	τ_2	
σ_2	1		1
+) σ_3	1	1	
$\sigma_2 + \sigma_3$	0	(1)	1

τ_2 is the pivot → $BT\ of\ \sigma_2 = 34 < 40 = BT\ of\ \tau_2$

→ $[34,\ 40)$ is PP

Assemble columns to reduce(ACTR)

- CTR(of $(d+1)$-dim)← {all $(d+1)$-cells} – {all d-pivots}

computational procedure

Read input data

↓

Make CTR (a list of 0-cell)

↓

compute pairs → Persistence pairs (0-dim)

CTR(of 1-cell)← (a list of 1-cells) – (pivots of 0-cell)

↓

compute pairs → Persistence pairs (1-dim)

CTR(of 2-cell)← (a list of 2-cells) – (pivots of 1-cell)

⋮

Compare with other software

- compare with DIPHA
    - data size : 200*200*200
    - the number of pairs : 39097
    - calculation time (file out) :

```
$ time ./dipha cf.complex cf.diagram

real    3m44.192s
user    3m21.078s
sys     0m16.328s
```

DIPHA

```
$ time ./cubicalripser_3dim
the number of pairs : 39097

real    0m51.945s
user    0m46.312s
sys     0m2.281s
```

Cubical Ripser