折り紙に現れる幾つかの幾何構造について

2017年10月21日

トポロジーとコンピュータ

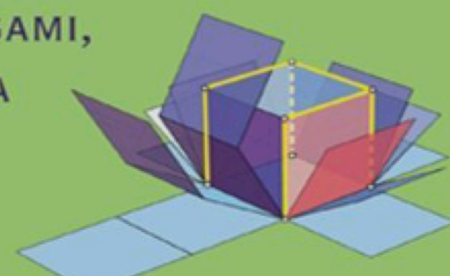
小林　毅

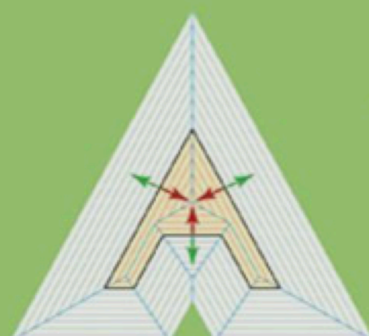
(奈良女子大学理学部)

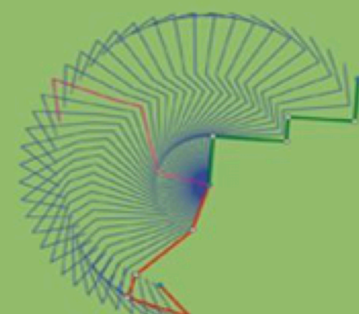
Prologue

HOW TO FOLD IT

THE MATHEMATICS OF LINKAGES, ORIGAMI, AND POLYHEDRA







JOSEPH O'ROURKE

CAMBRIDGE

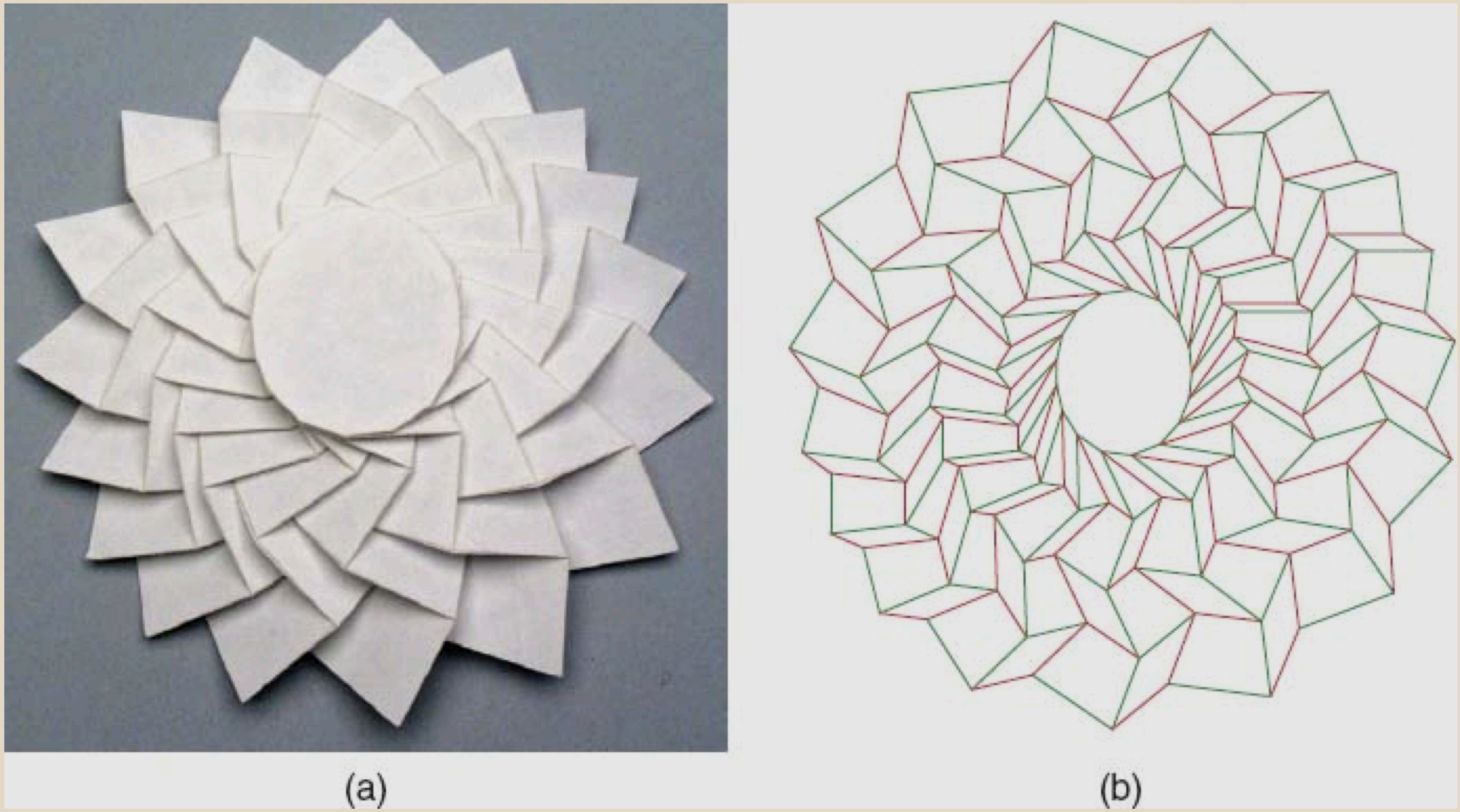


Figure 4.13. Robert Lang's *Oval Tessellation*, 1999.


NHK DVD
ポアンカレ予想
100年の格闘
数学者はキノコ狩りの夢を見る
「宇宙の形」の謎に迫る、
世紀の難問「ポアンカレ予想」とは?
100年もの時を経て
その証明についに成功した数学者は、
なぜ姿を消してしまったのか?
ひとつの難問にとりつかれた数学者たちの
一世紀にわたる闘いの物語。
DVD VIDEO

Thurston proposed Geomtrization Conjecture

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 6, Number 3, May 1982

THREE DIMENSIONAL MANIFOLDS, KLEINIAN GROUPS AND HYPERBOLIC GEOMETRY

BY WILLIAM P. THURSTON

1. A conjectural picture of 3-manifolds. A major thrust of mathematics in the late 19th century, in which Poincaré had a large role, was the uniformization theory for Riemann surfaces: that every conformal structure on a closed oriented surface is represented by a Riemannian metric of constant curvature. For the typical case of negative Euler characteristic (genus greater than 1) such a metric gives a hyperbolic structure: any small neighborhood in the surface is isometric to a neighborhood in the hyperbolic plane, and the surface itself is the quotient of the hyperbolic plane by a discrete group of motions. The exceptional cases, the sphere and the torus, have spherical and Eu[illegible] structures.

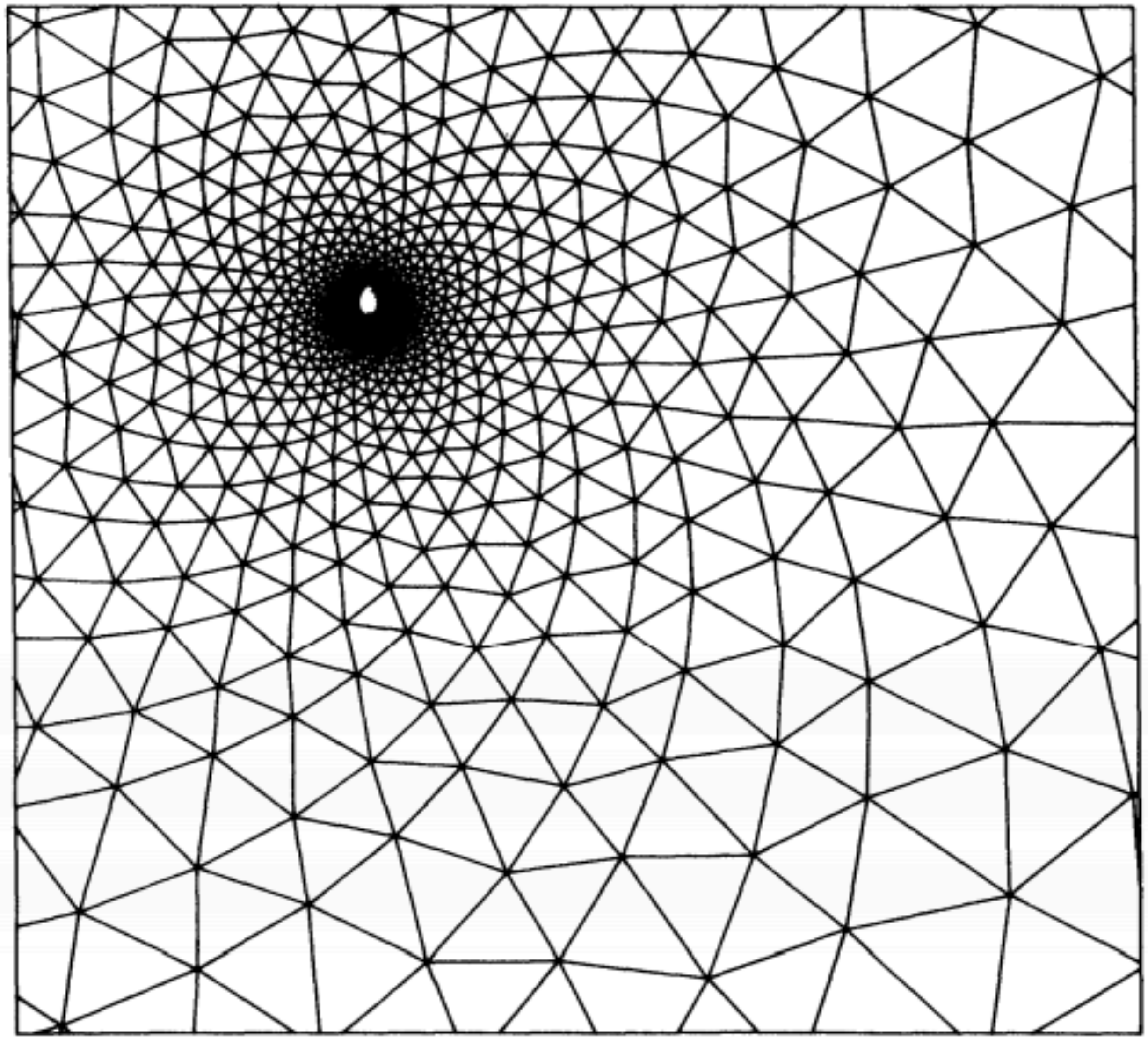
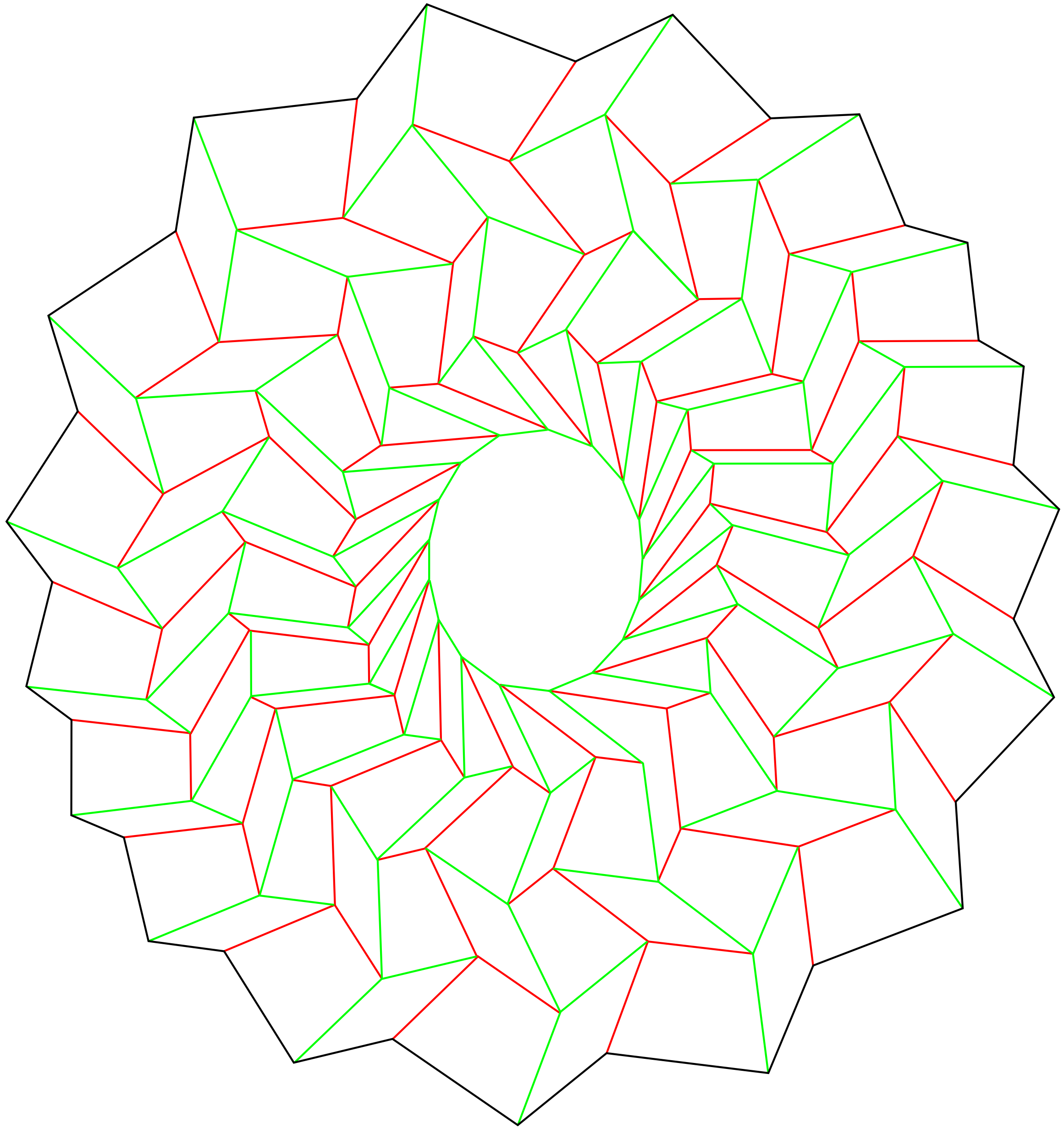


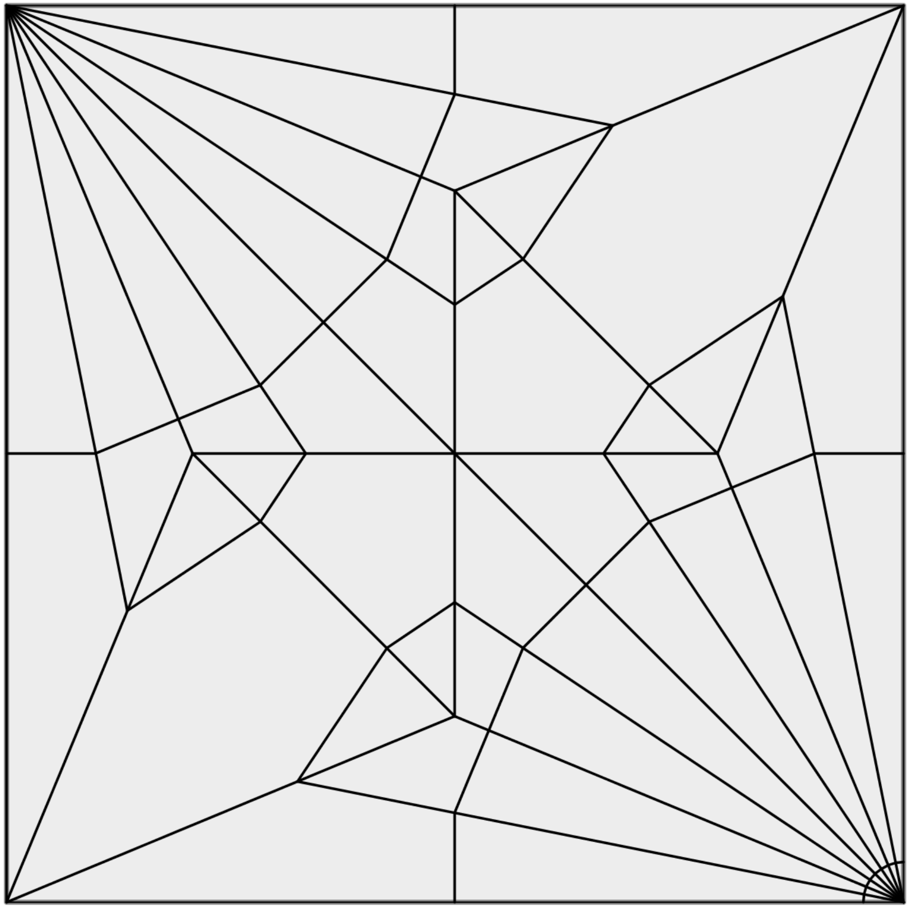
FIGURE 4. Three o'clock sky.

Oval Tessellation

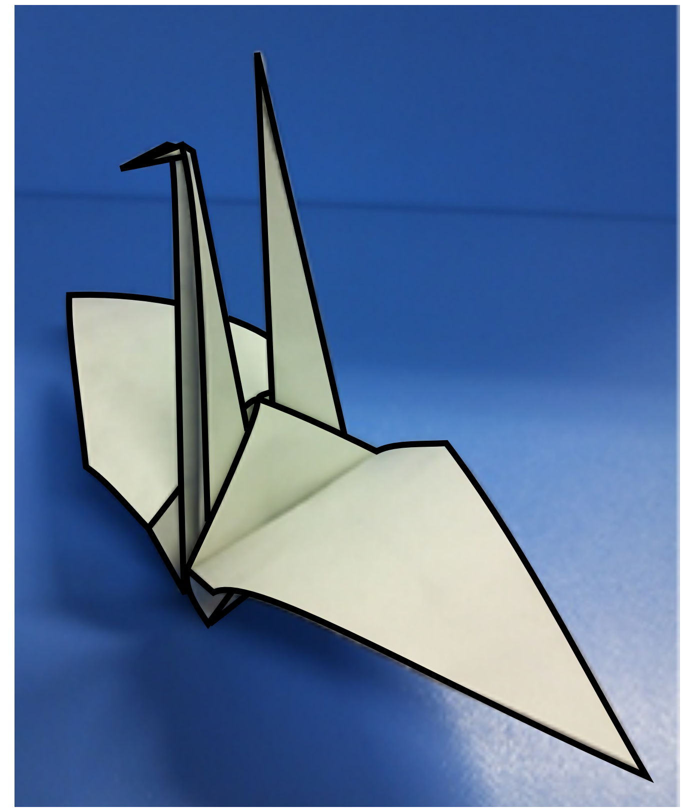


Mathematical Formulation

Origami

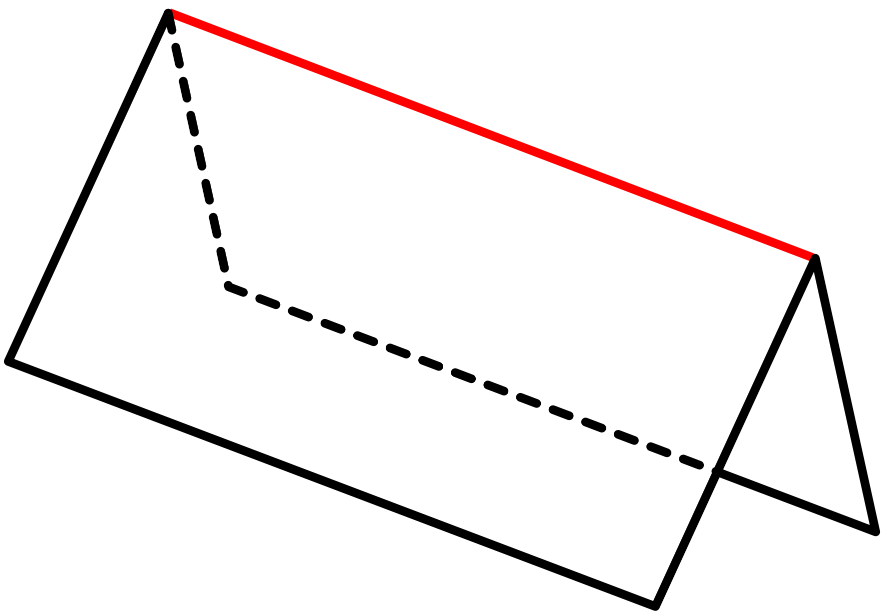


Origami paper

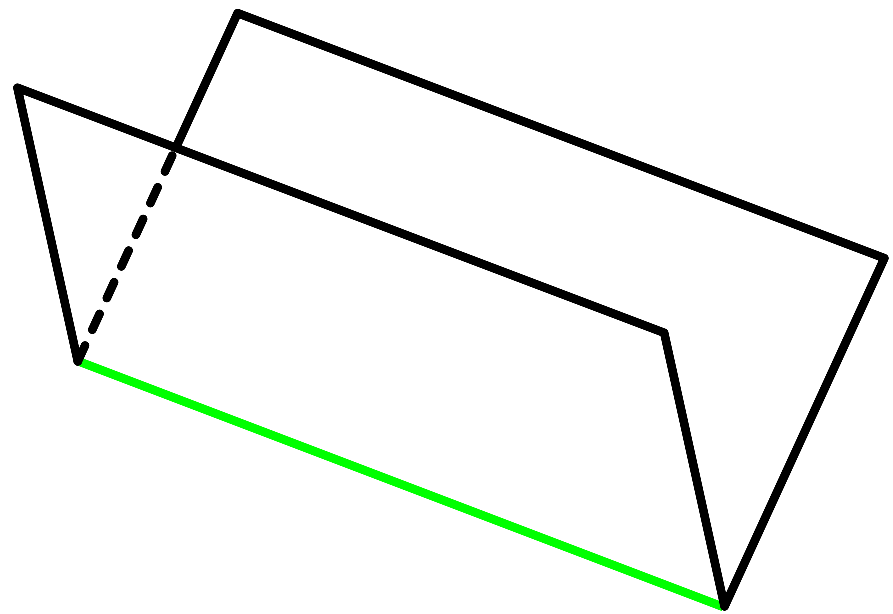


Origami

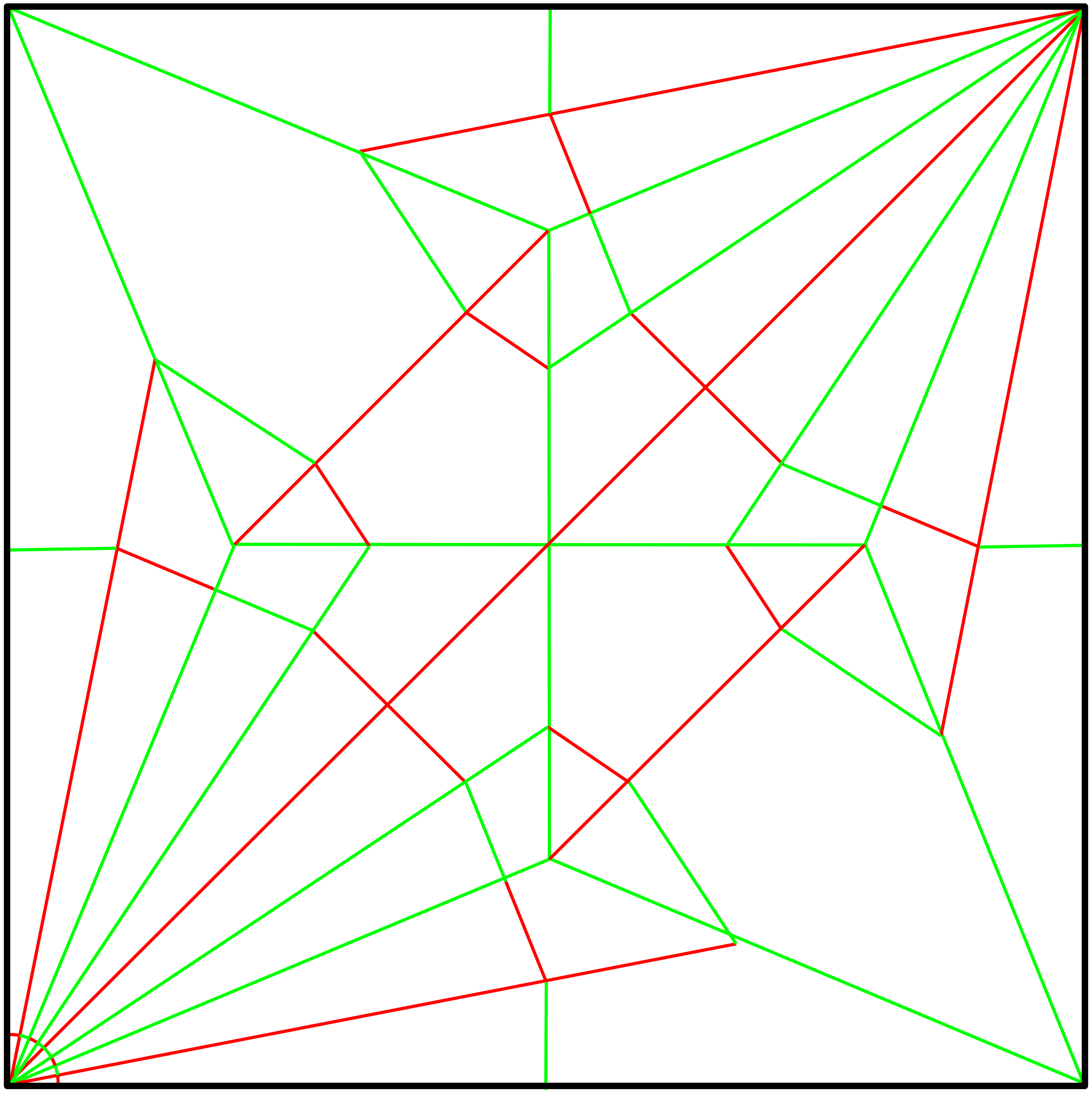
Fold line（折り線）



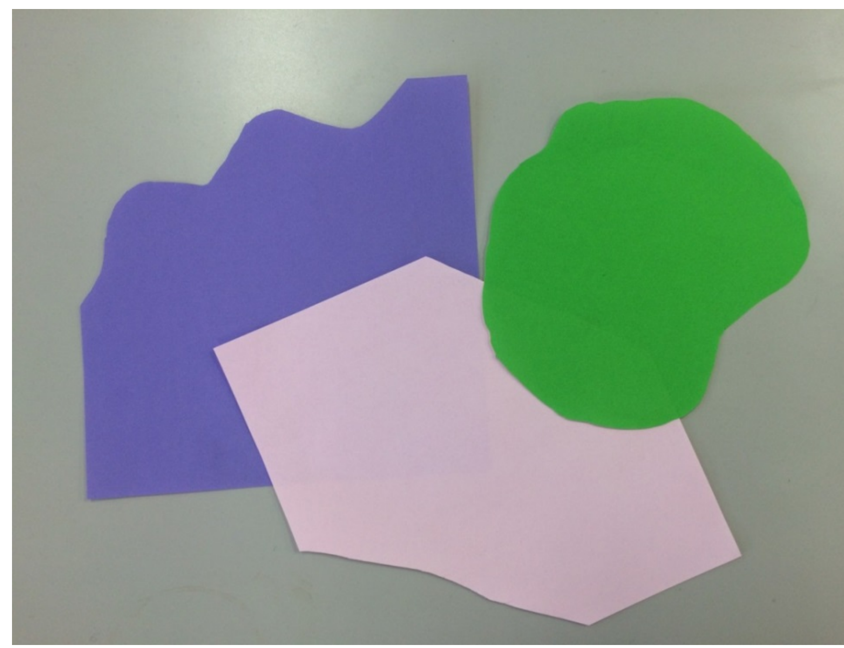
Mountain fold



Valley fold



Crease pattern（展開図）

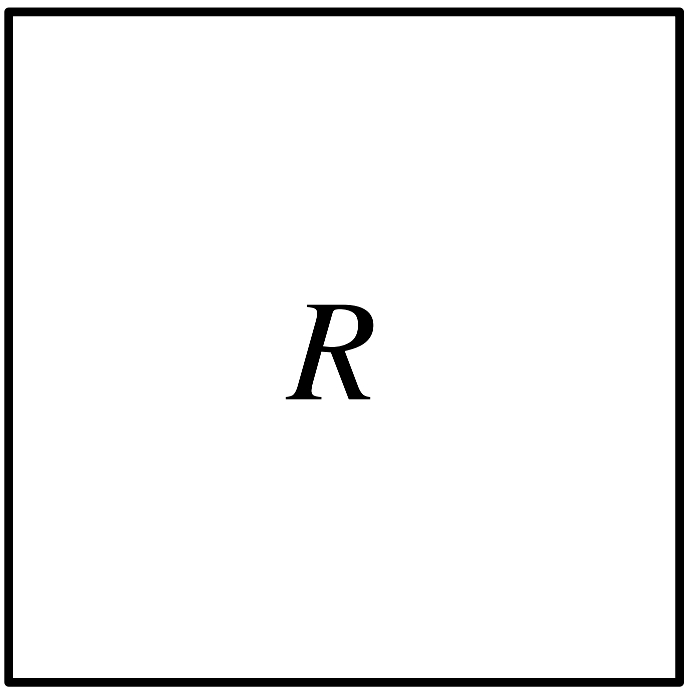


Origami paper is a region R in R^2

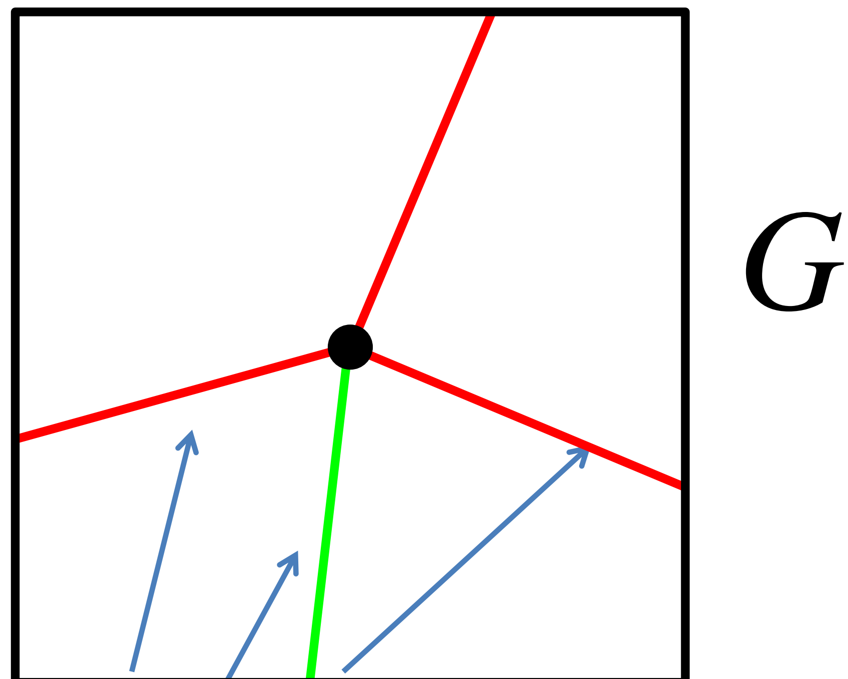
$$R \subset R^2$$

Mathematical Formulation

$R \subset R^2$: a region

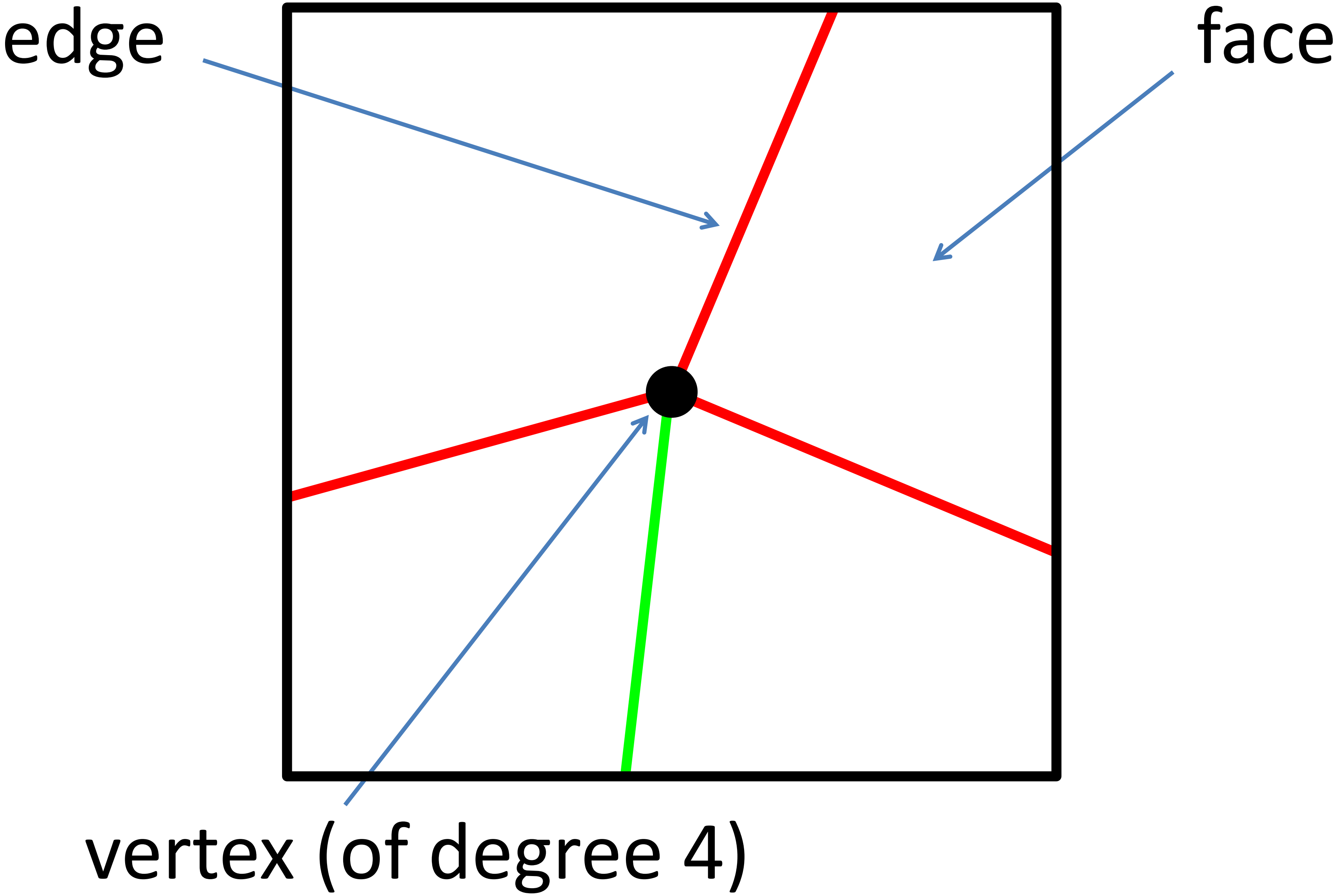


Origami paper



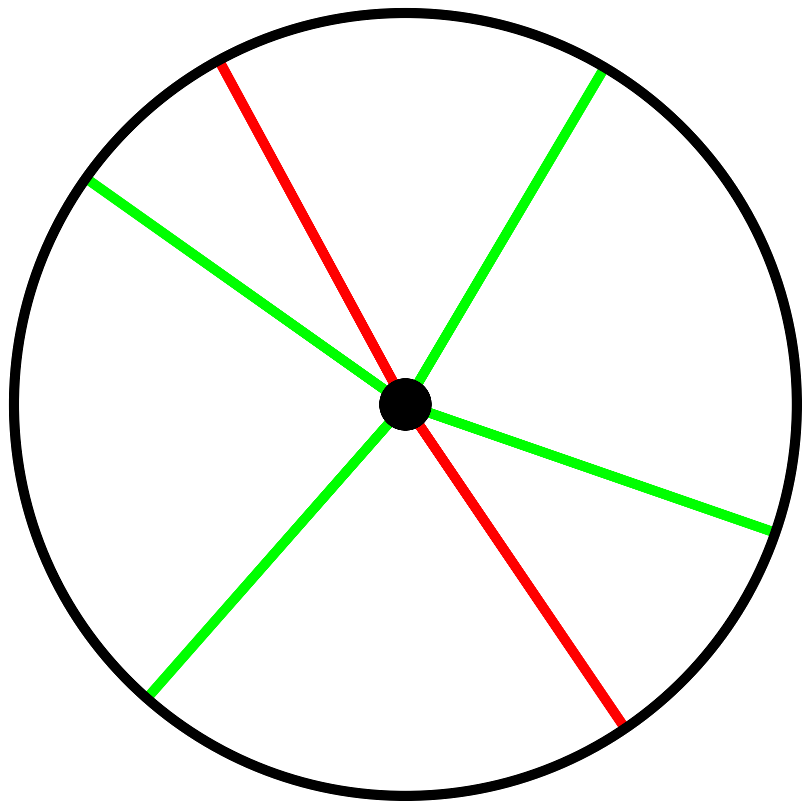
fold line（折り線）

crease pattern（展開図）

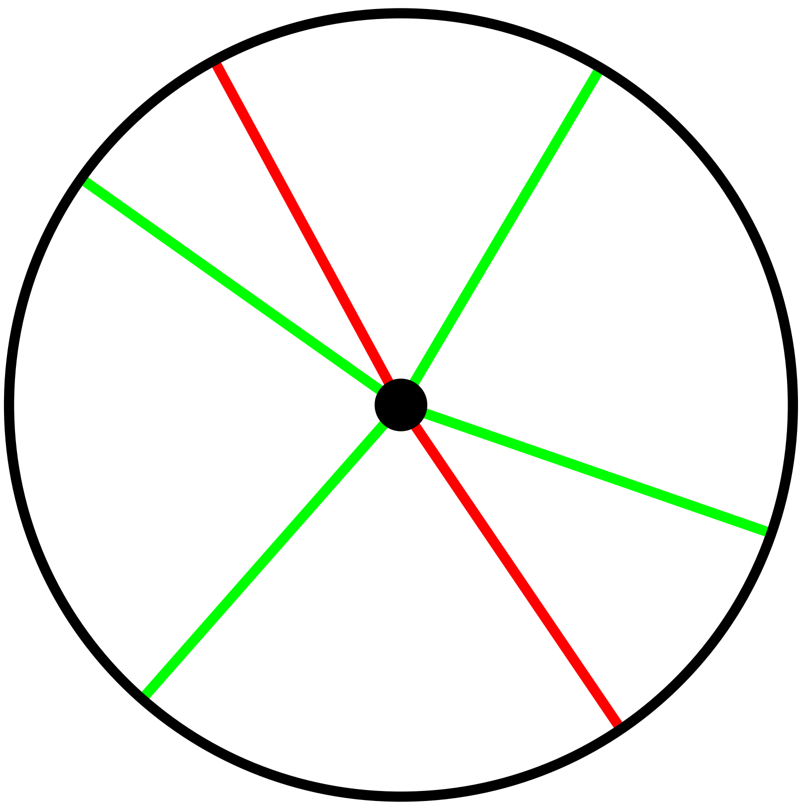

edge
face
vertex (of degree 4)

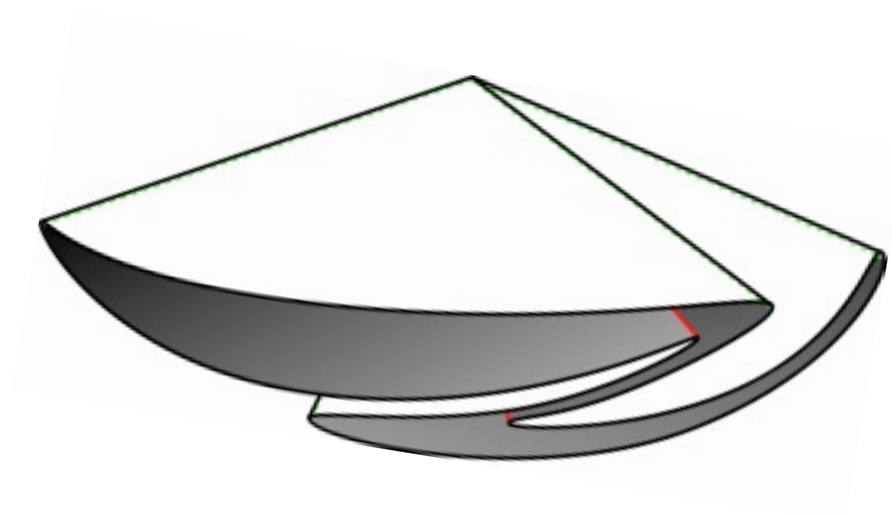
One-vertex flat folding

One vertex folding



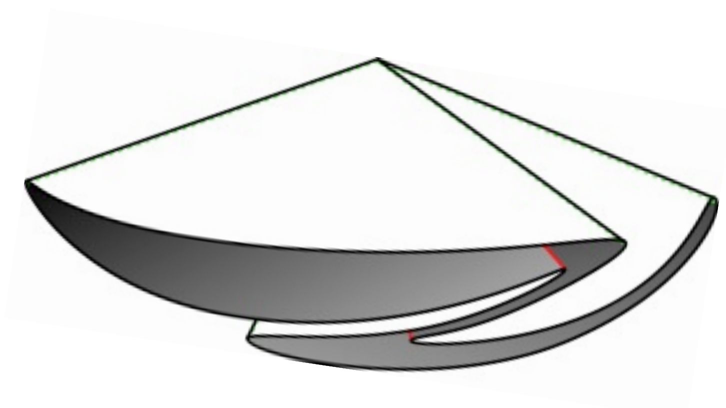
Flat foldable crease pattern

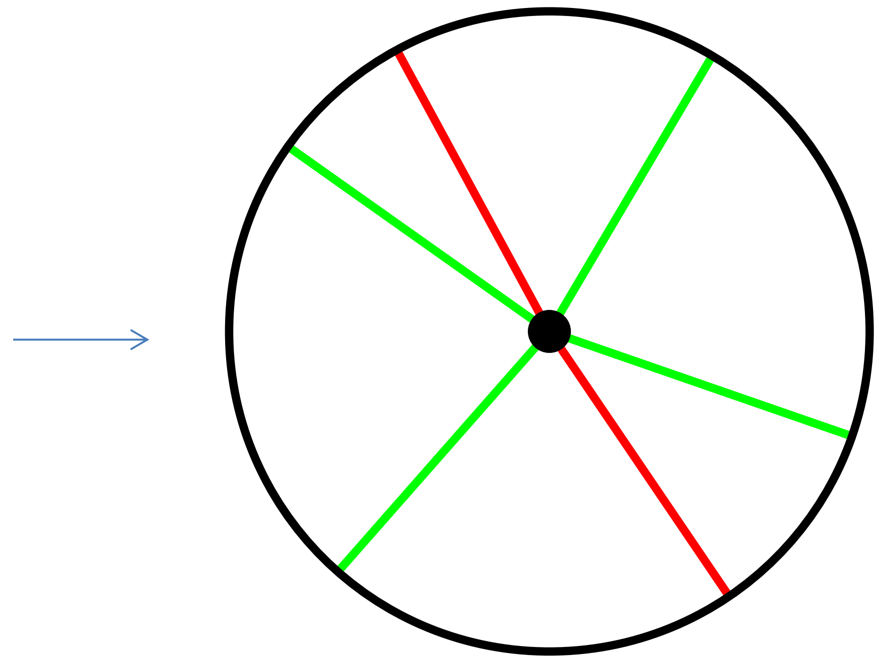




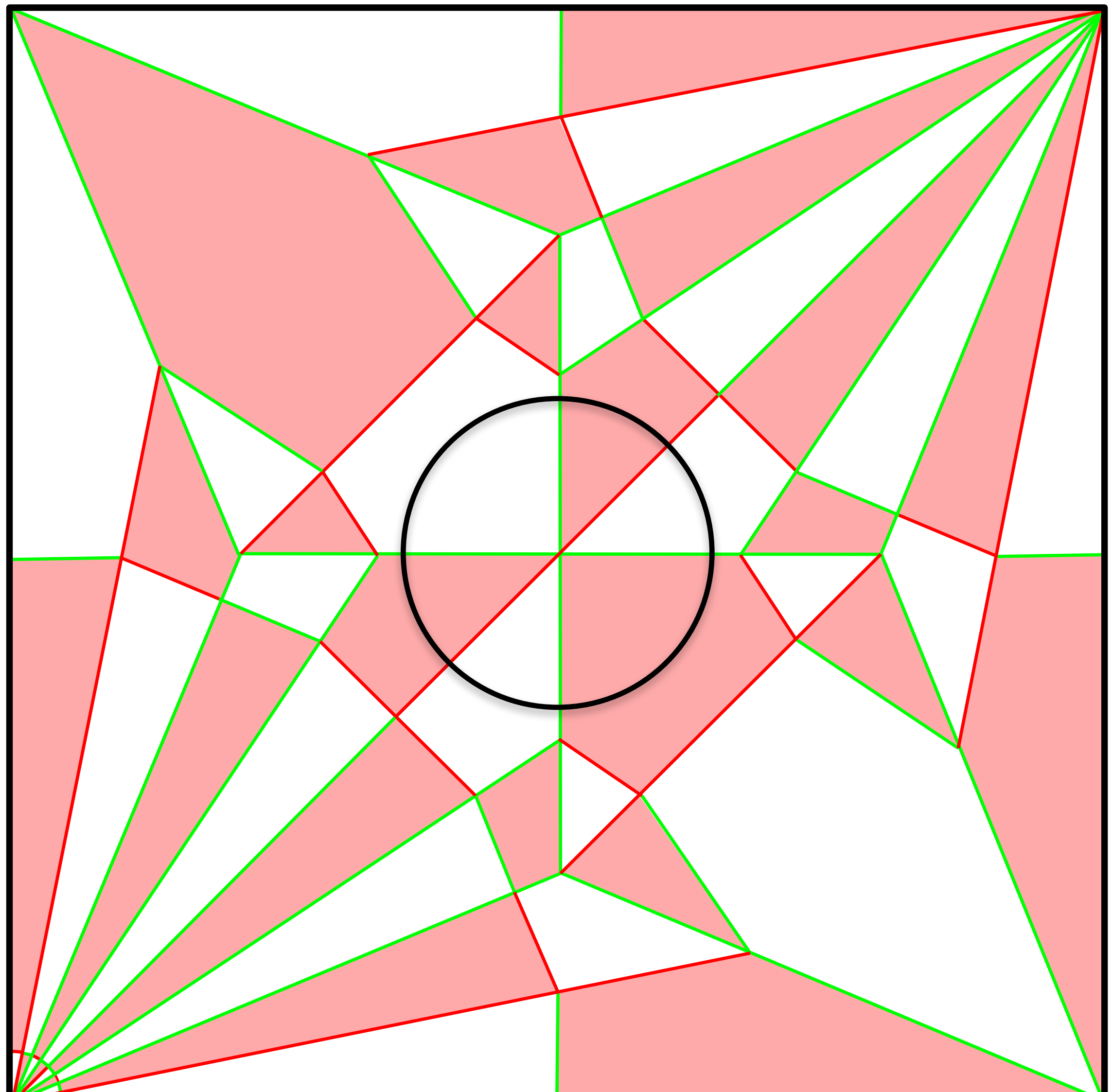
Even degree

If one vertex folding (R, G) is flat, then the degree of the vertex is even.

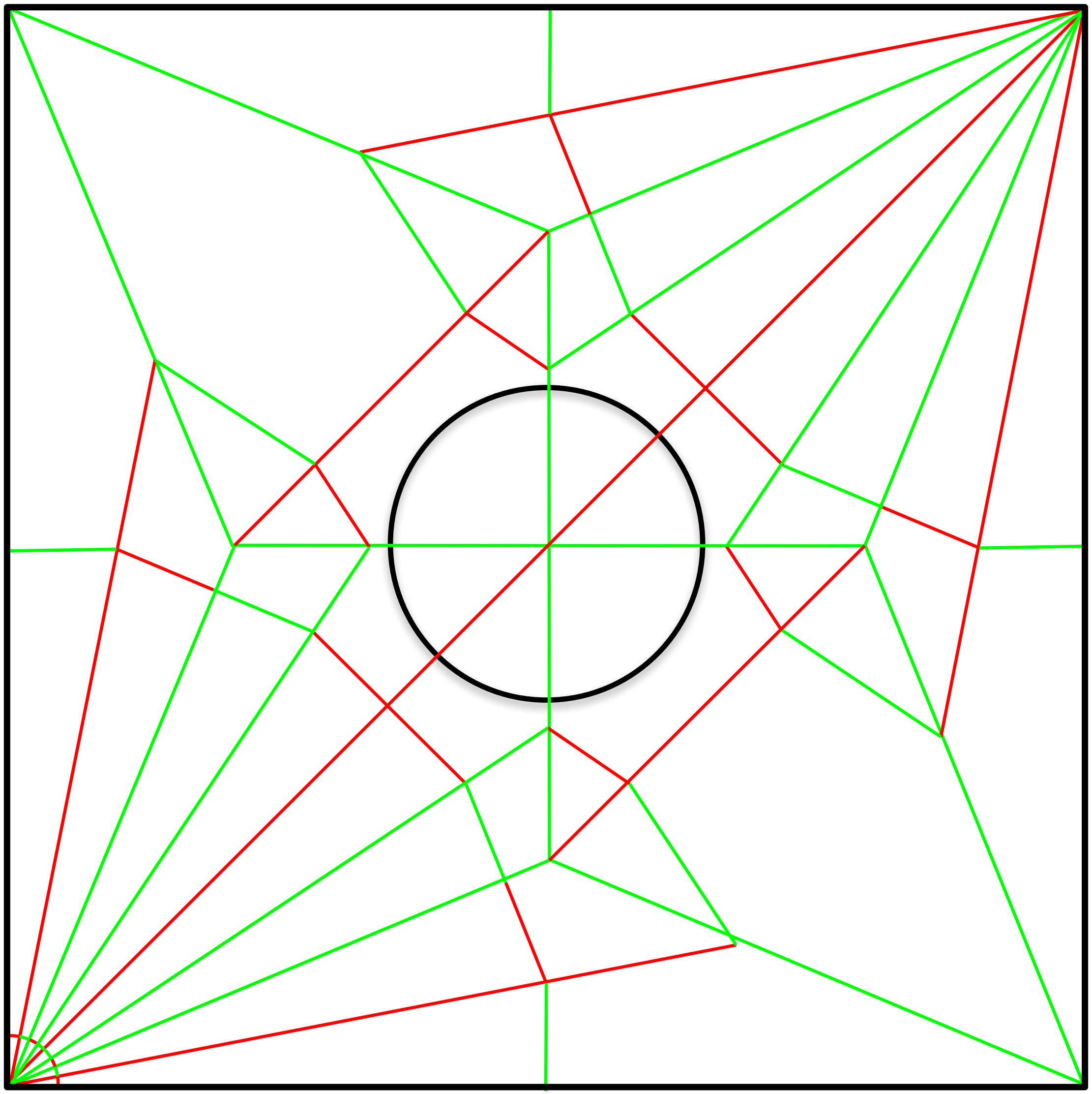




Even degree

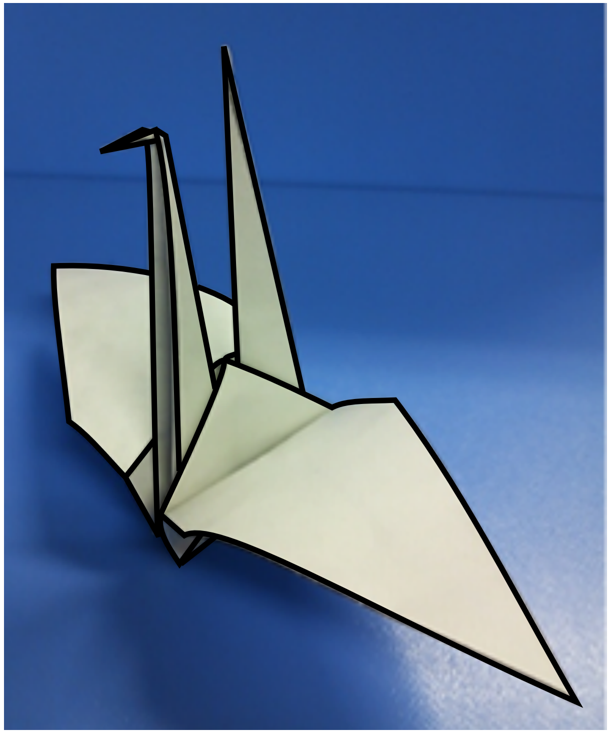


Mountain-Valley counting

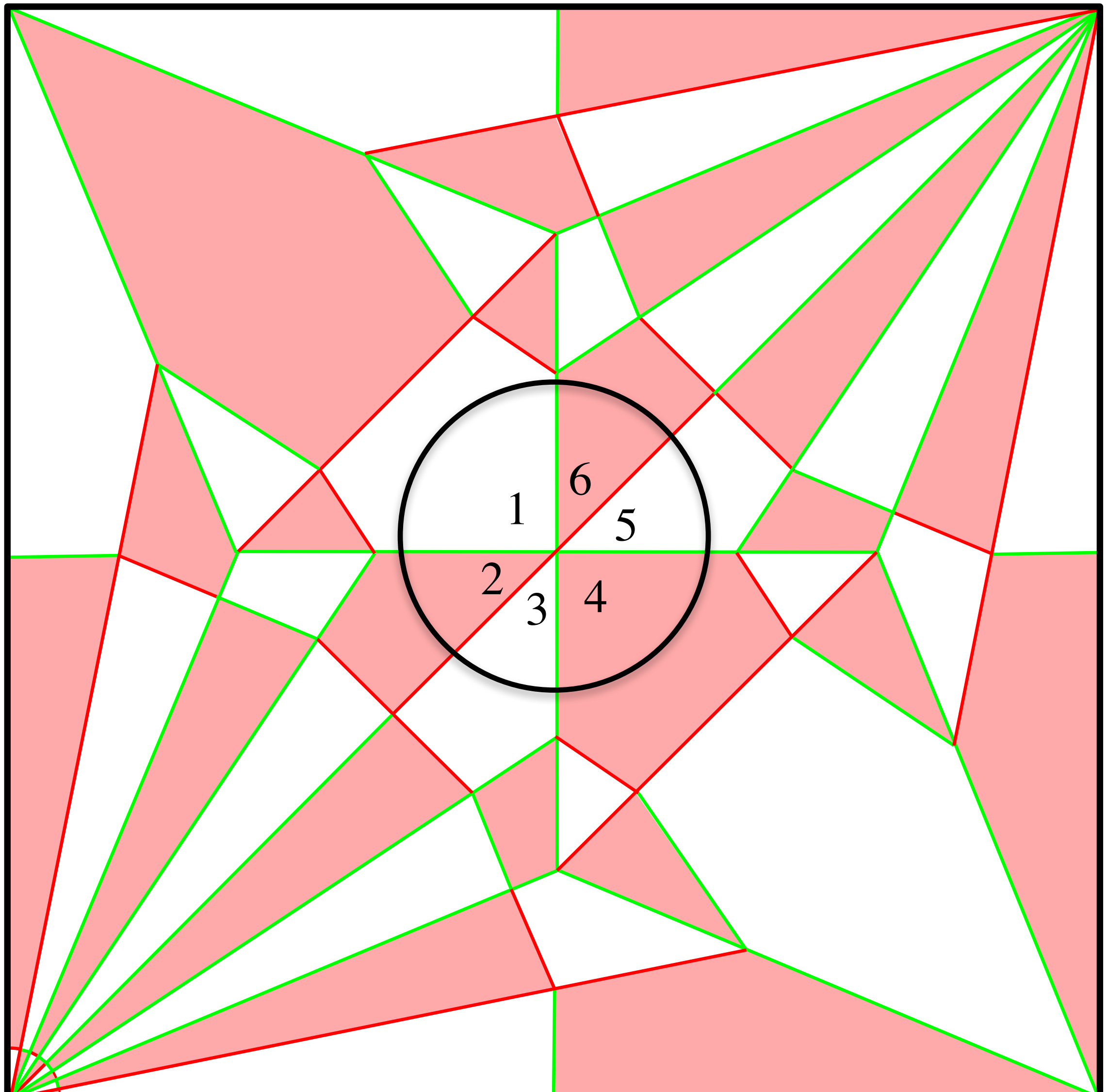


(Maekawa-Justin)

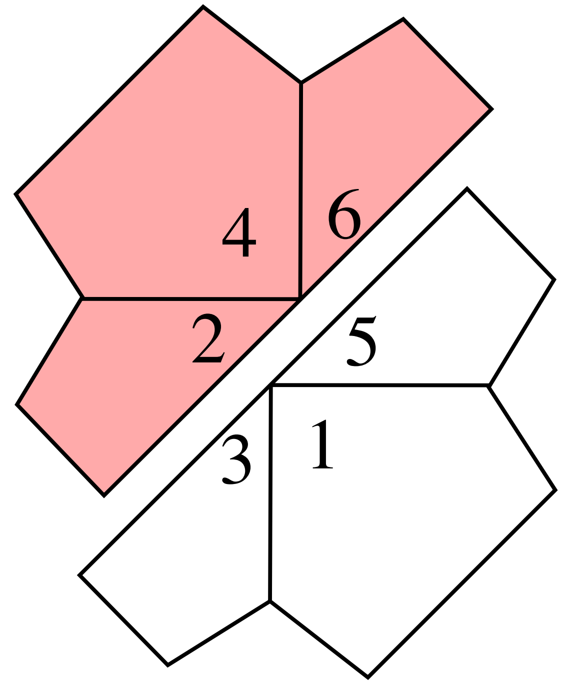
$$M - V = \pm 2$$



Angles around a vertex



Kawasaki–Justin



Theorem (degree 4 flat folding)

The crease pattern is flat foldable

Iff

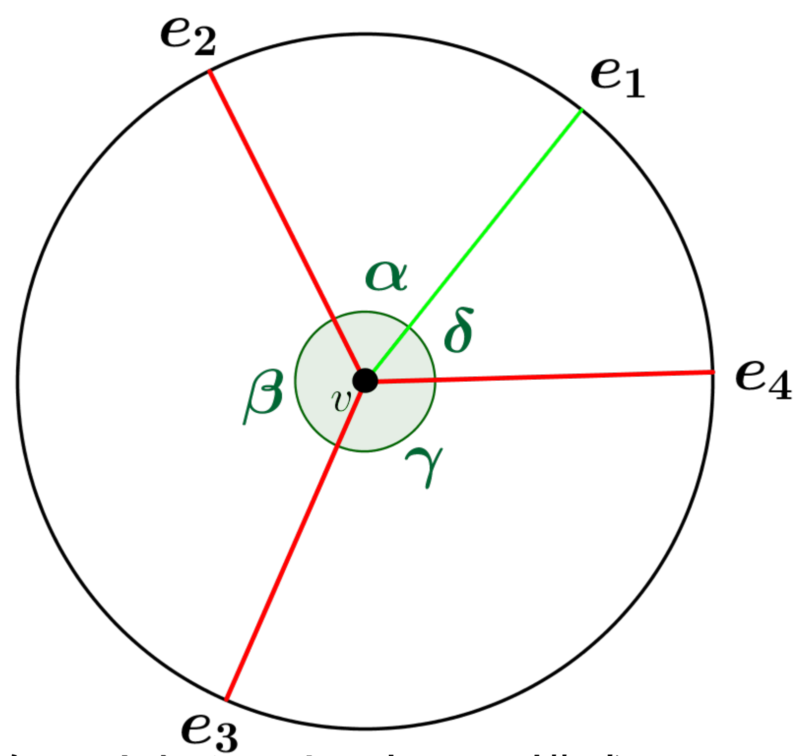
1.(Maekawa-Justin)

$(M, V) = (3,1)$ or $(1,3)$

2.(Kawasaki-Justin)

$\alpha + \gamma = \beta + \delta = \pi$

3. If e_1 is the “exceptional” edge, then $\alpha \leq \beta$



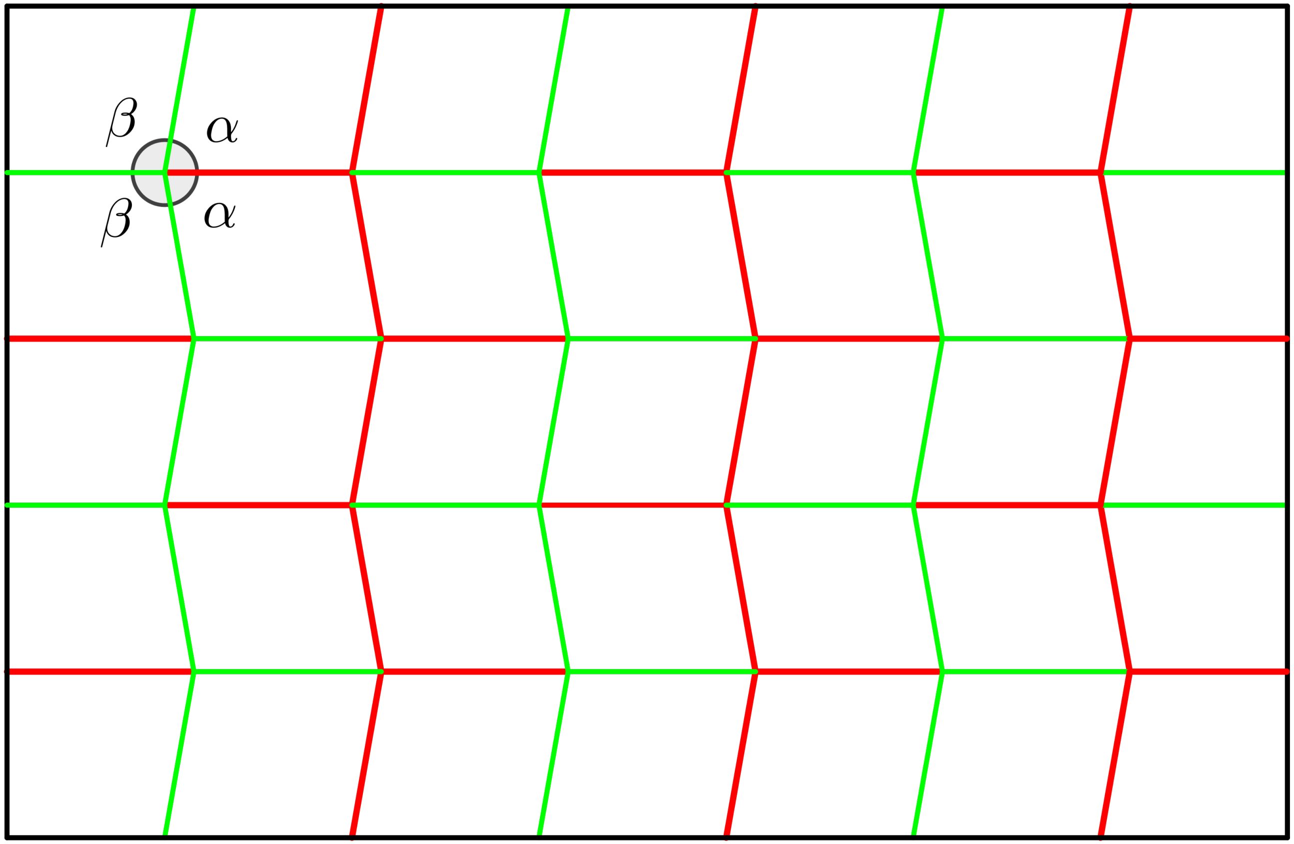
入井美紀：2次元トーラスの相似構造による，一般化されたミウラ折りの構成：奈良女子大学修士論文

After finishing the paper we found Fushimi-Fushimi, and Murata had already obtained the same result.

Fushimi, K. and Fushimi, (1979). Origami No Kikagaku (Geometry of Origami), Nihon Hyoronsha

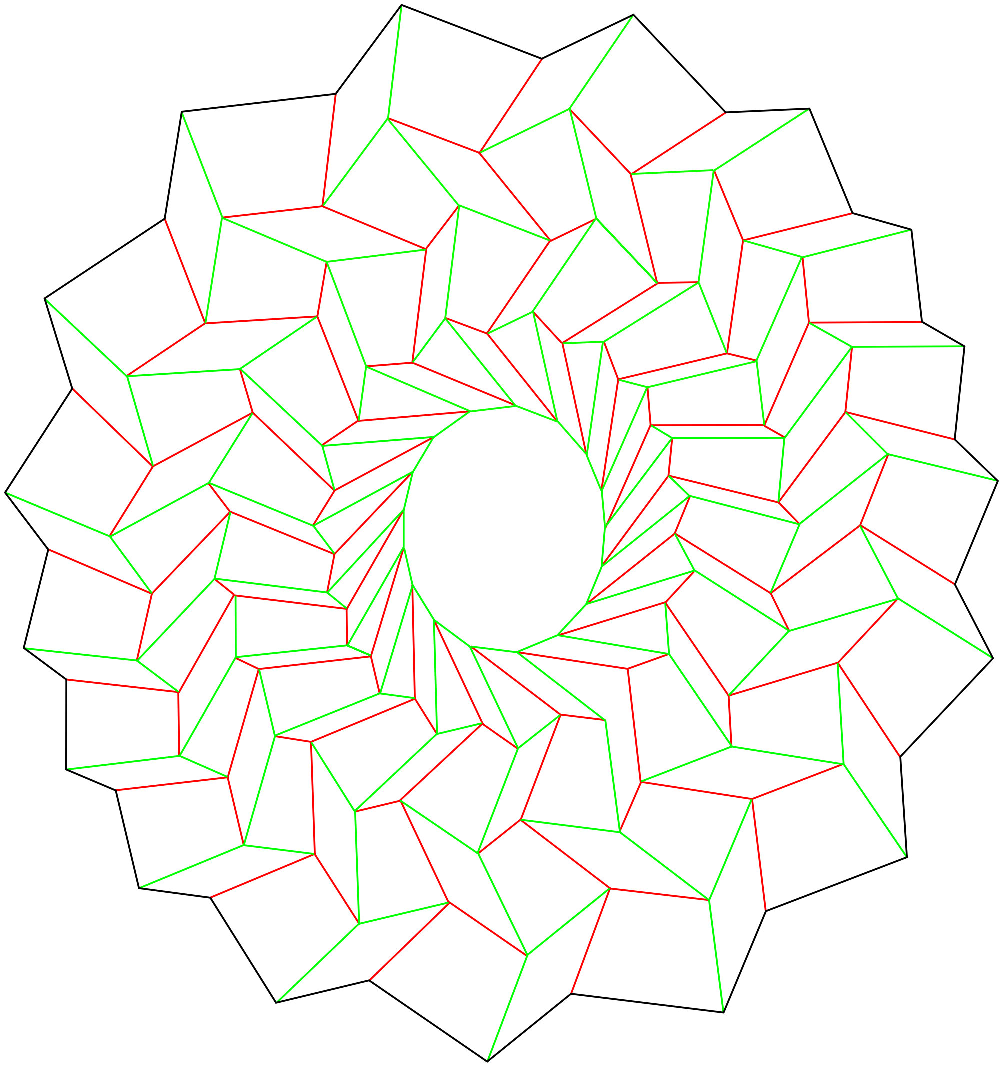
Murata, S., The theory of paper sculpture, Bulletin of Junior College of Art, 1966, Vol.4, 61-66, http://ci.nii.ac.jp/naid/110004714036/

Miura Folding



$\alpha = \alpha$, $\beta = \beta$,
$\gamma = \beta$, $\delta = \alpha$

Oval Tessellation



First construction:
Similarity Str. on 2-dim. torus

(G,X)-structure

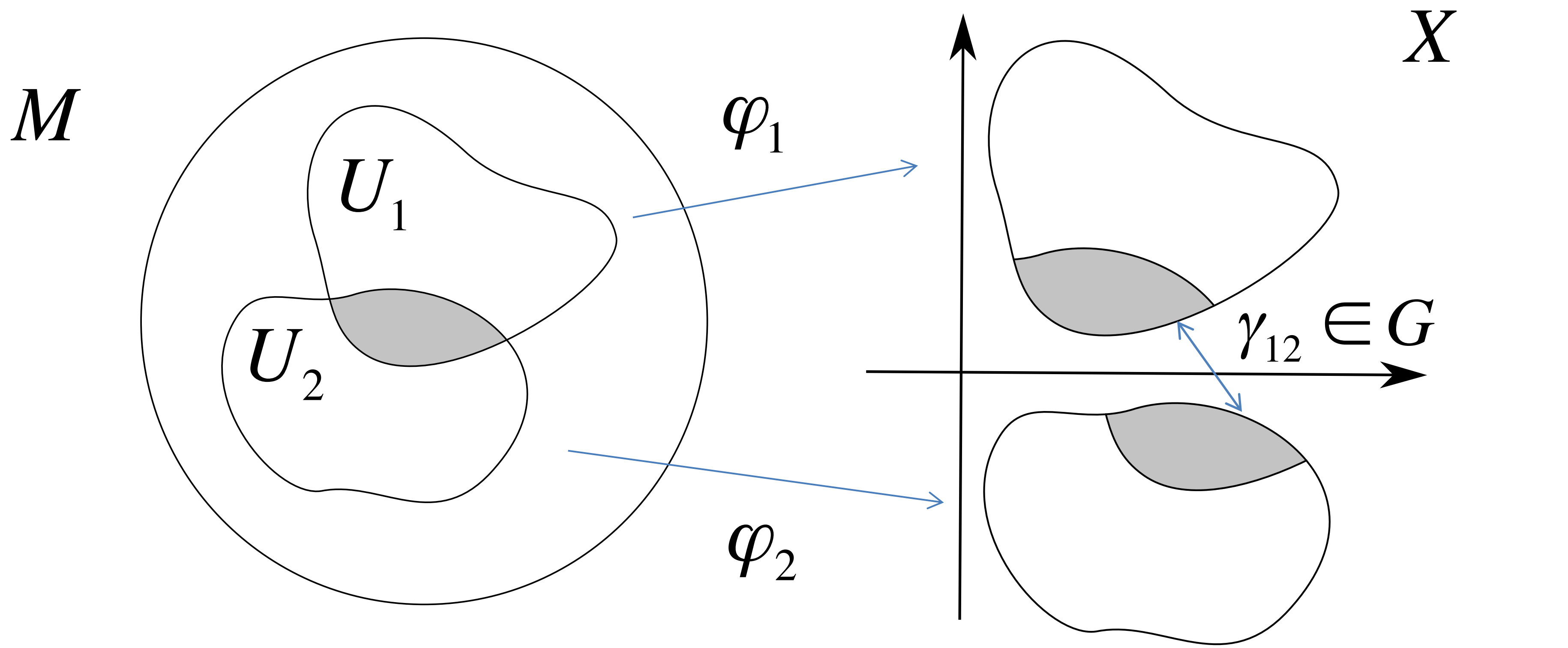
Thurston Lecture Note: Chapter 3

Let X be any real analytic manifold, and

G a group of real analytic diffeomorphisms of X.

M is (G, X)-manifold if:

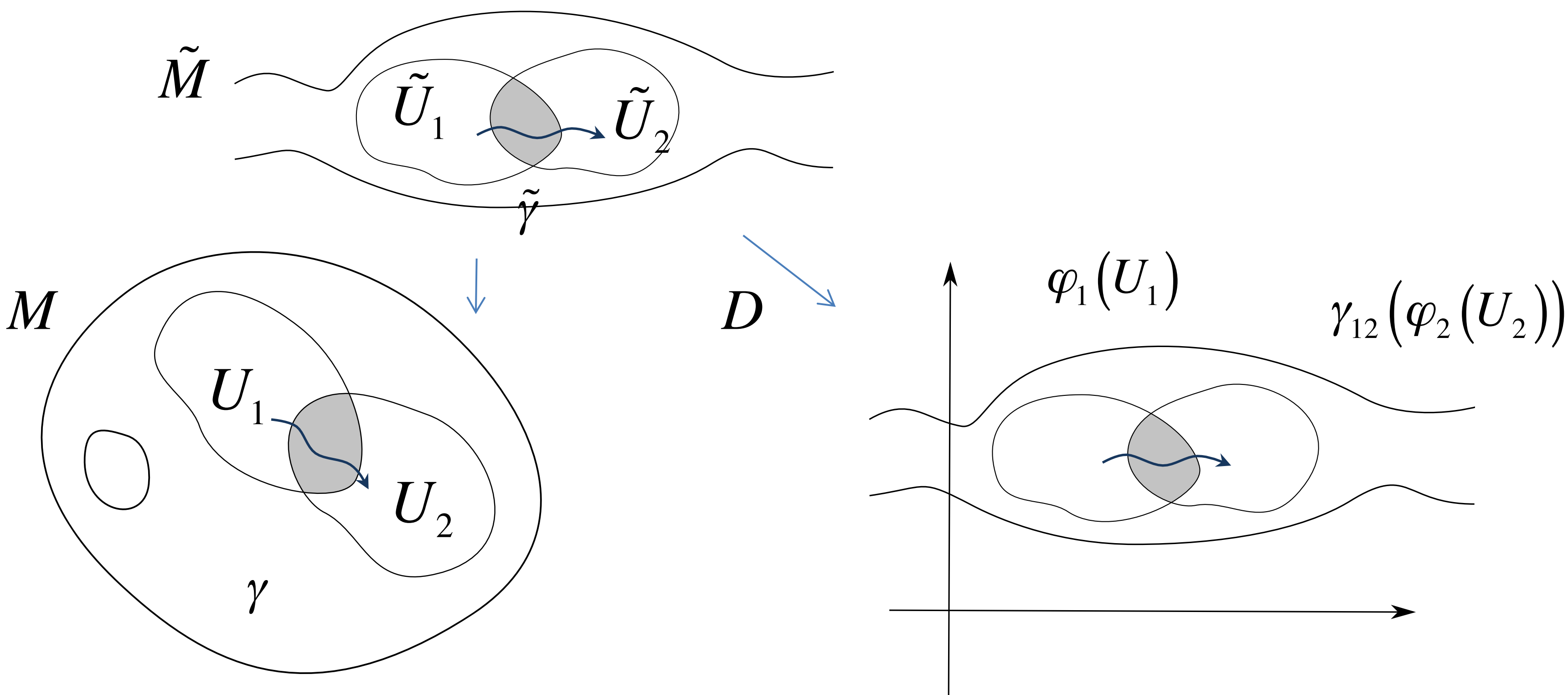
There exist U1, U2, . . . coordinate charts for M, with maps $\phi_i : U_i \to X$ and transition functions γ_{ij} in G satisfying $\gamma_{ij} \circ \phi_i = \phi_j$



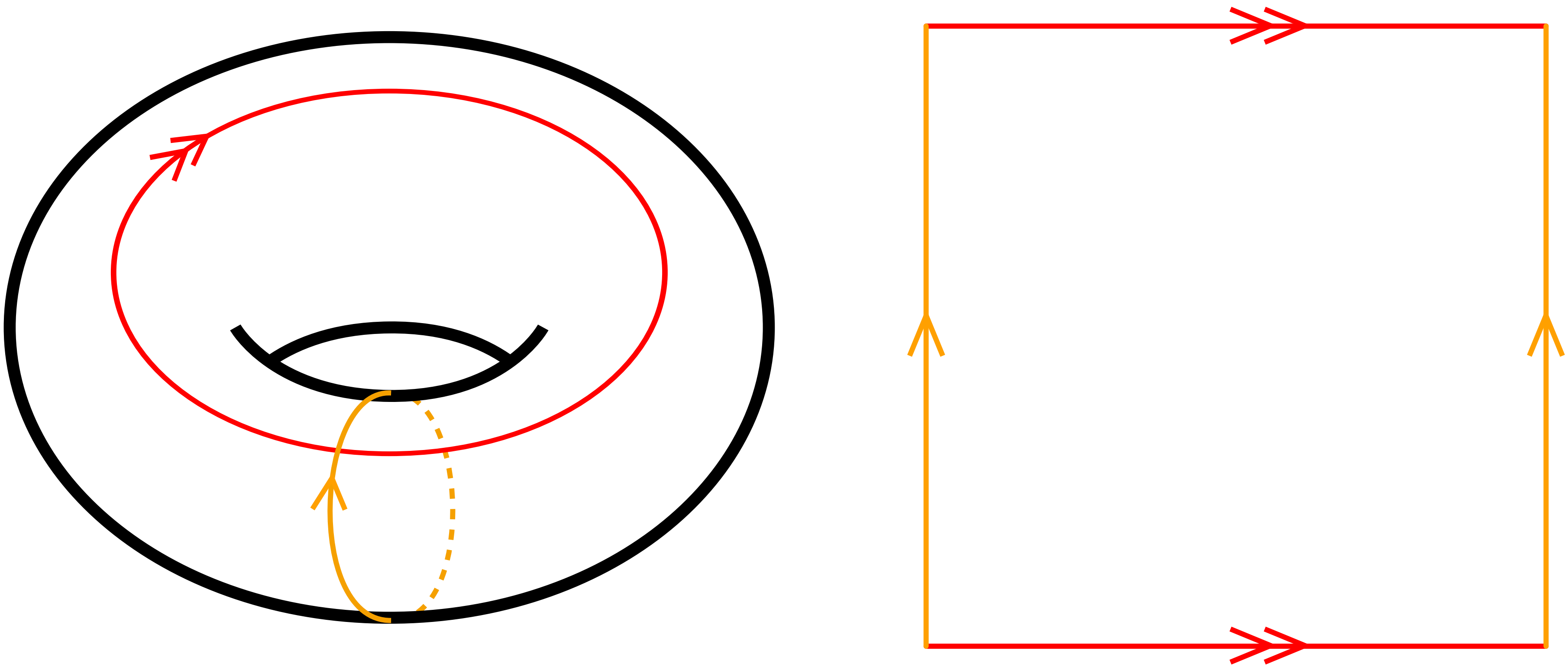
Developing map

Consider an analytic continuation of ϕ_1 along a path γ in M beginning in U_1 ⇒ There is a global analytic continuation of ϕ_1 on the universal cover of M.

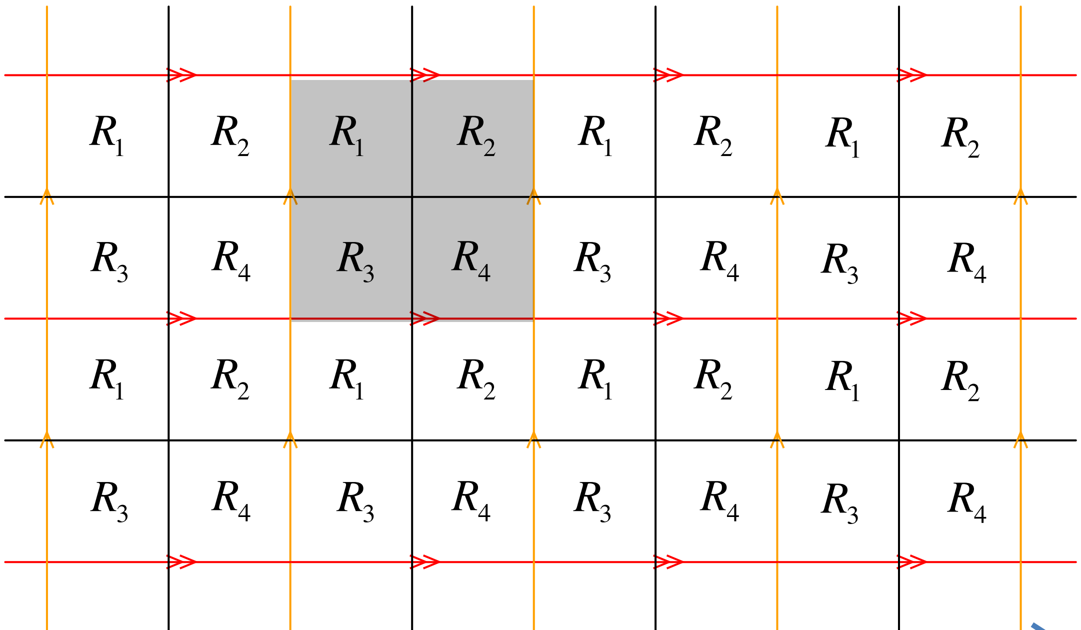
This map, D : M˜ → X, is called the developing map.



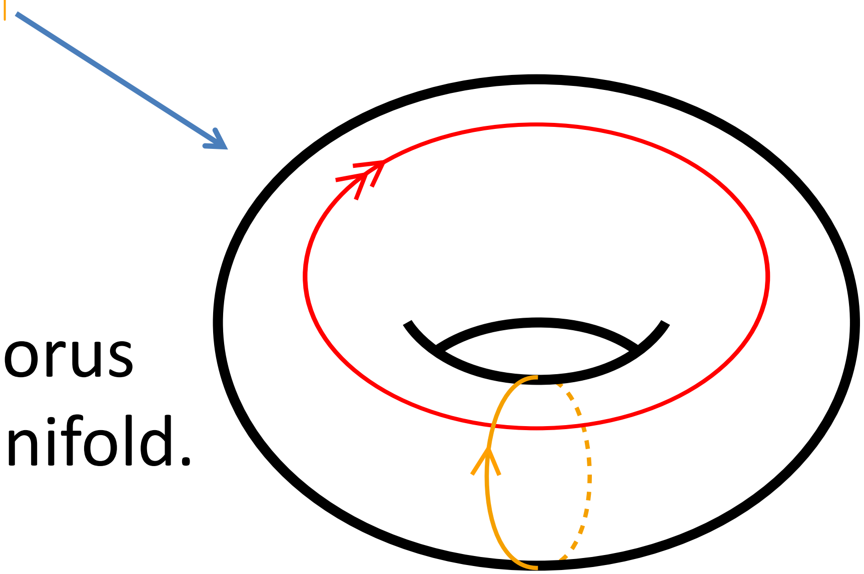
Euclidean str. on 2-dim. torus



Developing map



Universal cover of the torus



This figure shows that the 2-dim torus
Is a $(E^2, \mathrm{Isom}(E^2))$ (: Euclidean) manifold.

Similarity Structure（相似構造）

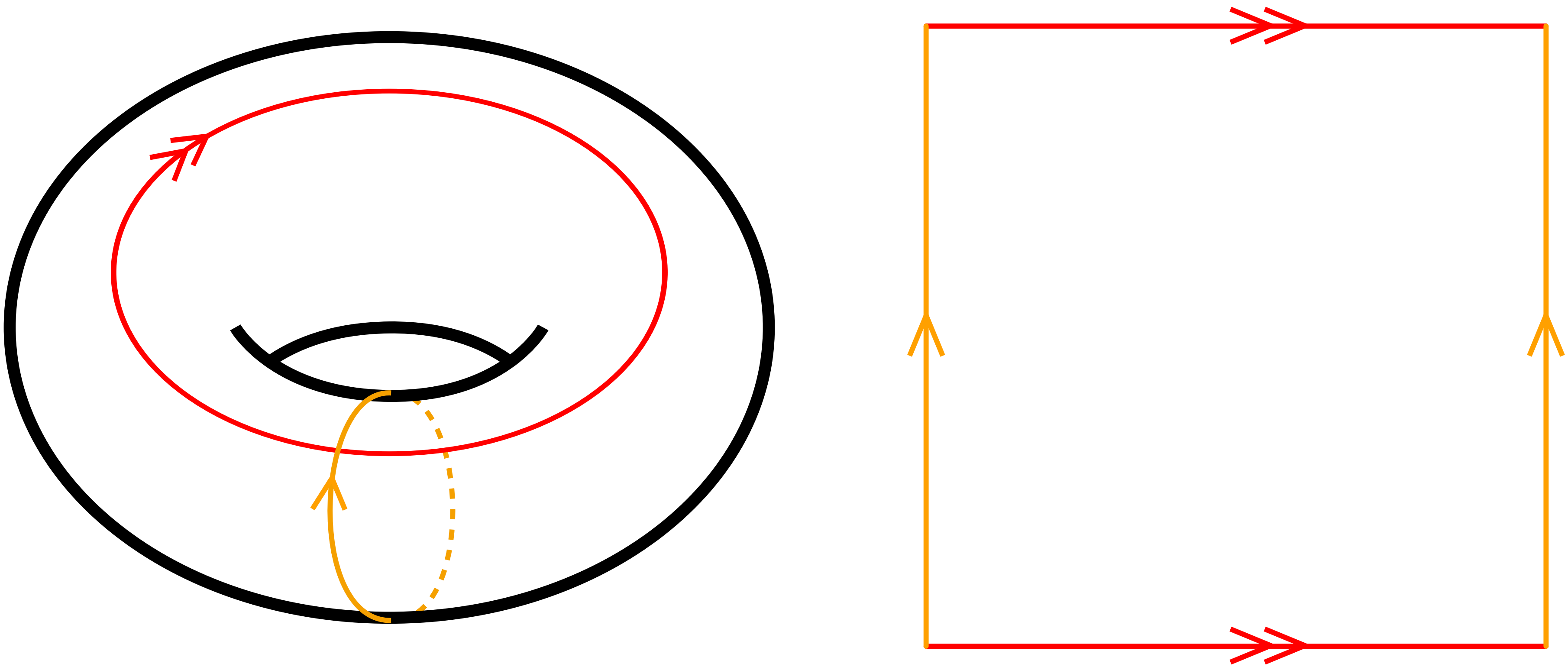
(G, X) -structure with

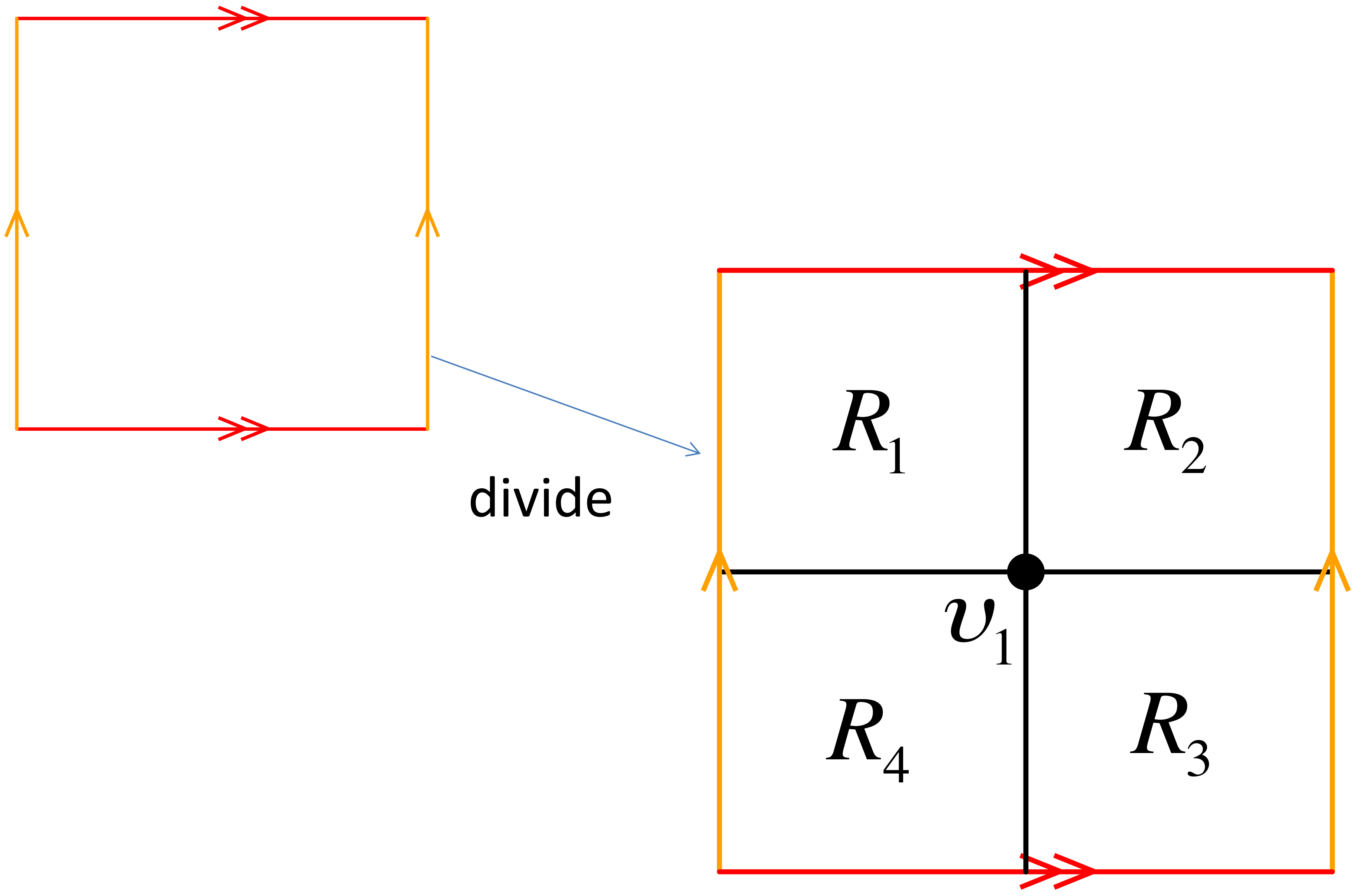
X : Euclidean plane

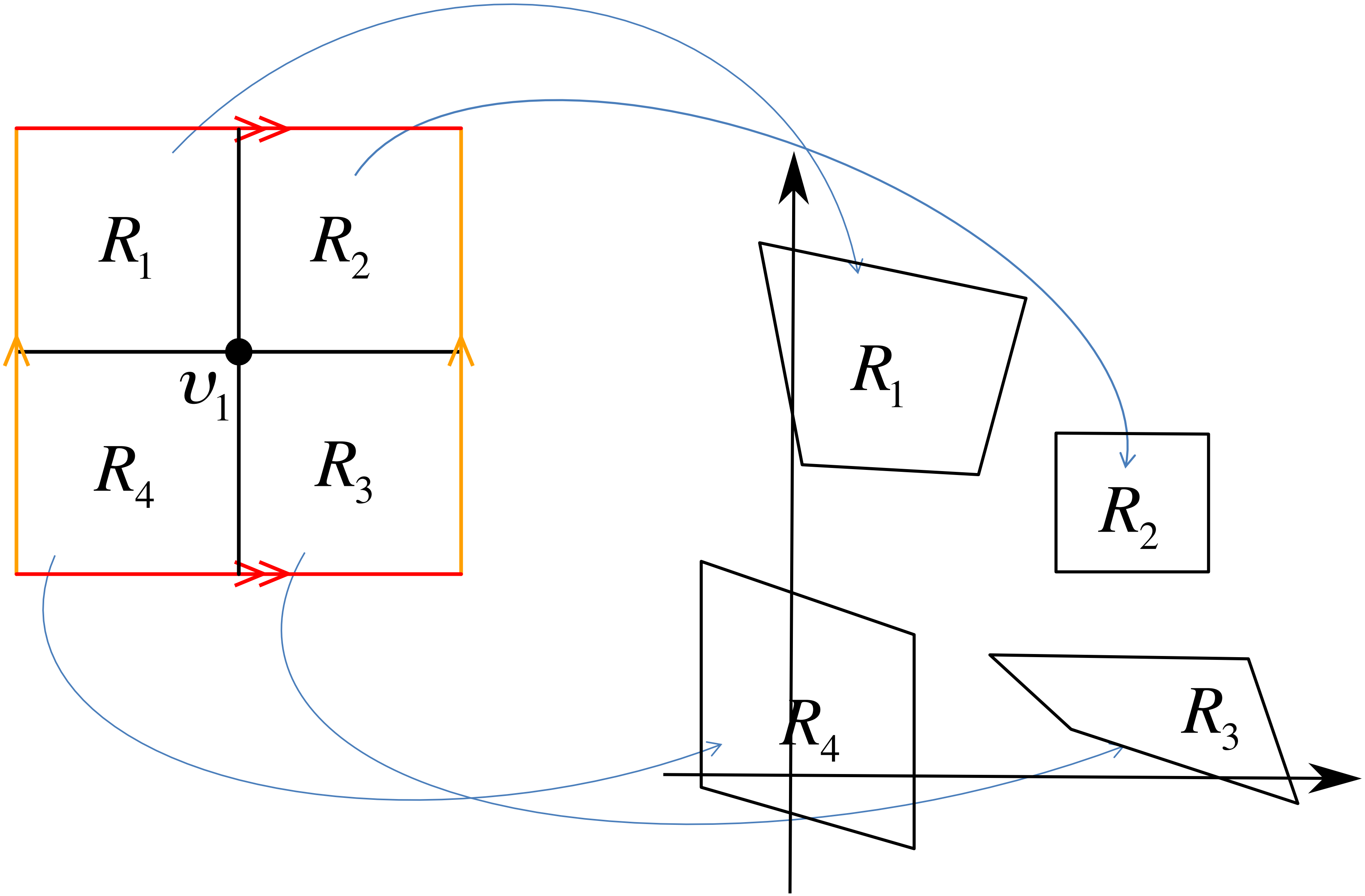
G : the group of similar translations on X

is (2-dim.) similarity str.

Similarity Structure on 2-dim. torus

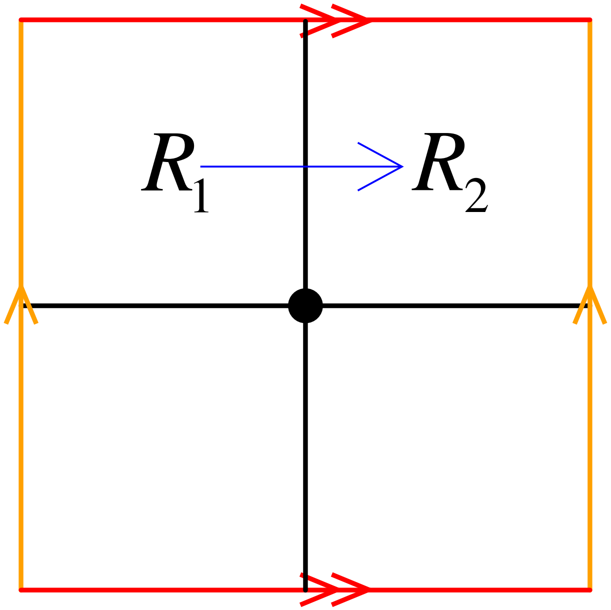


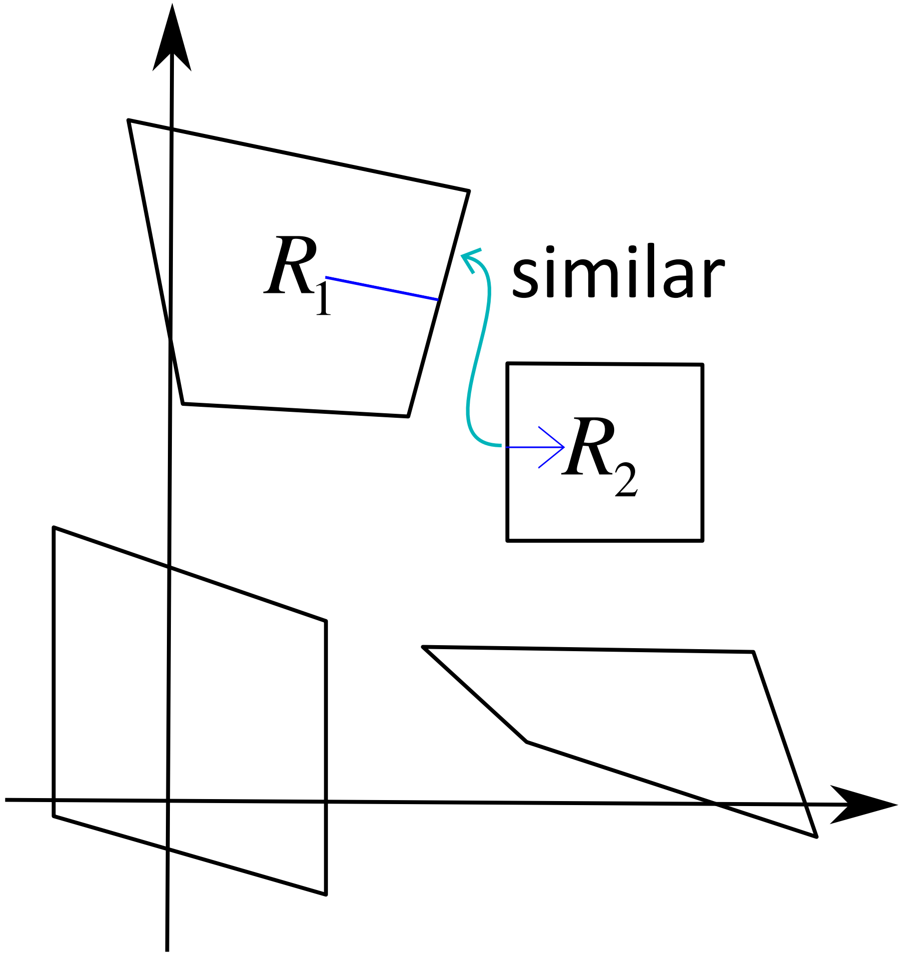

divide
R_1
R_2
R_3
R_4
v_1



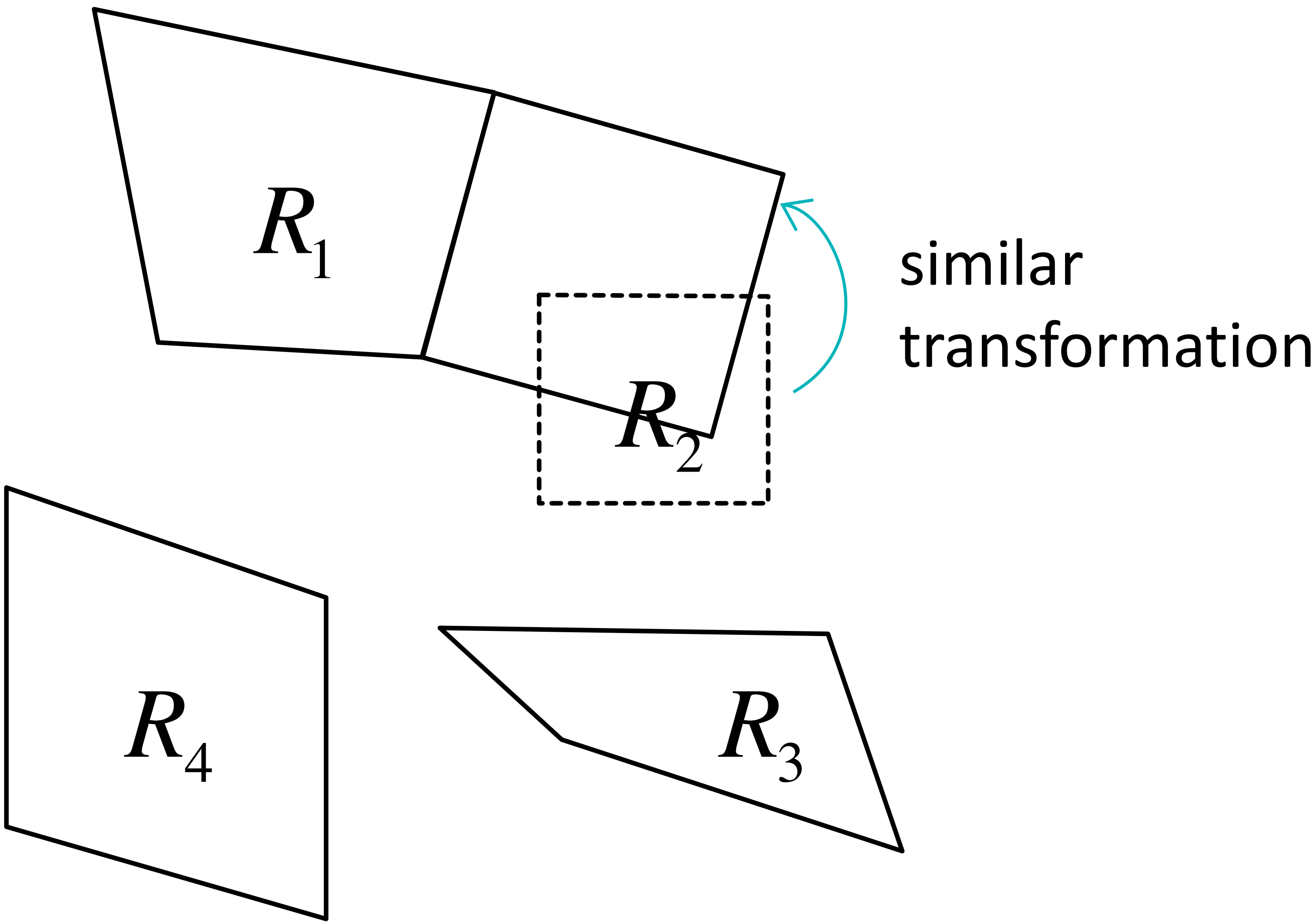
R_1
R_2
v_1
R_4
R_3
R_1
R_2
R_4
R_3

Consistency condition

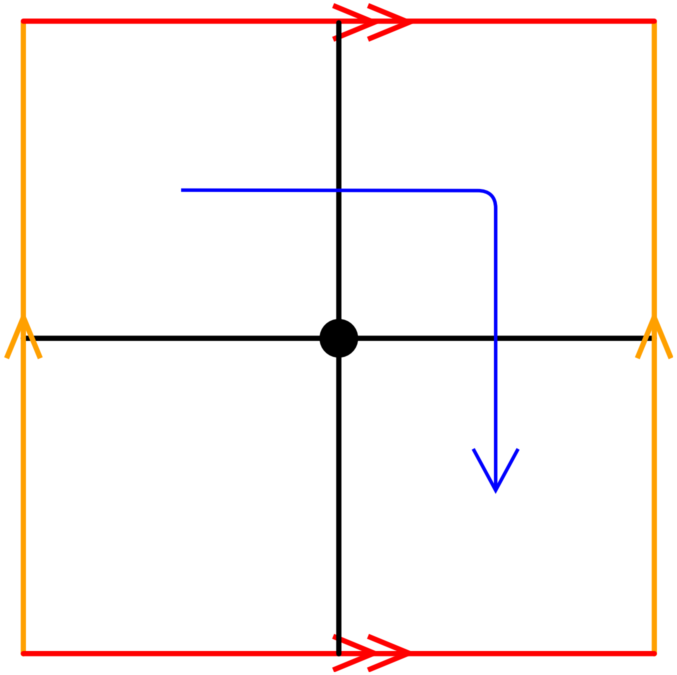


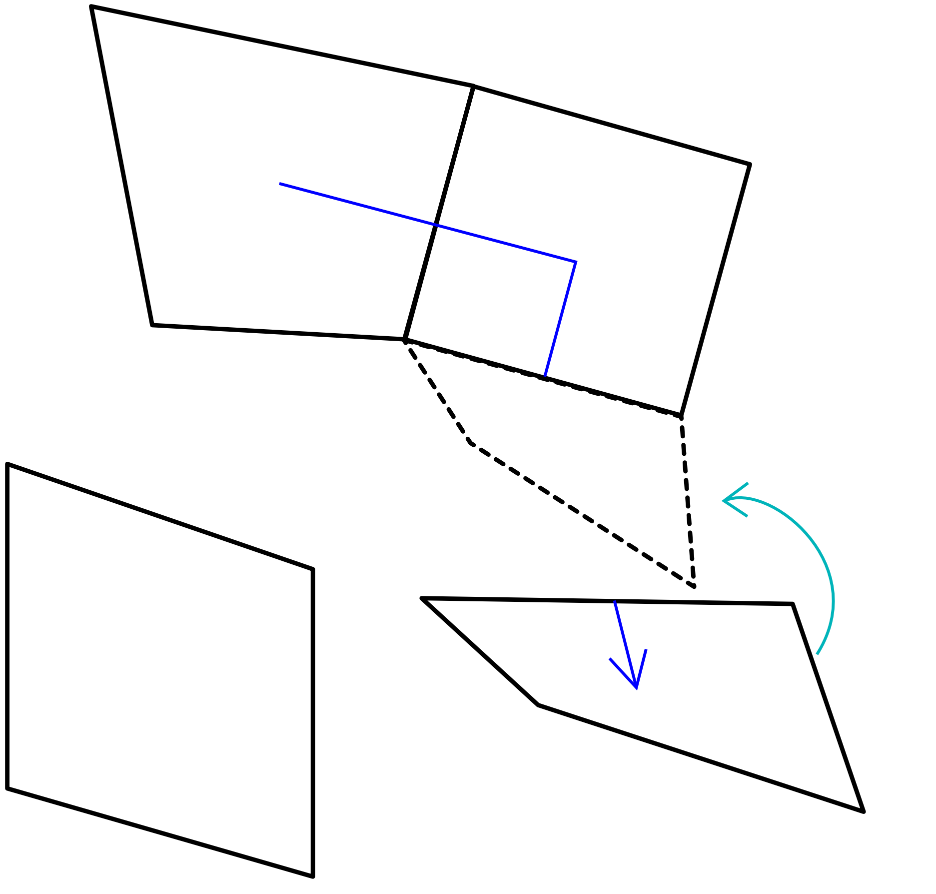


Consistency condition

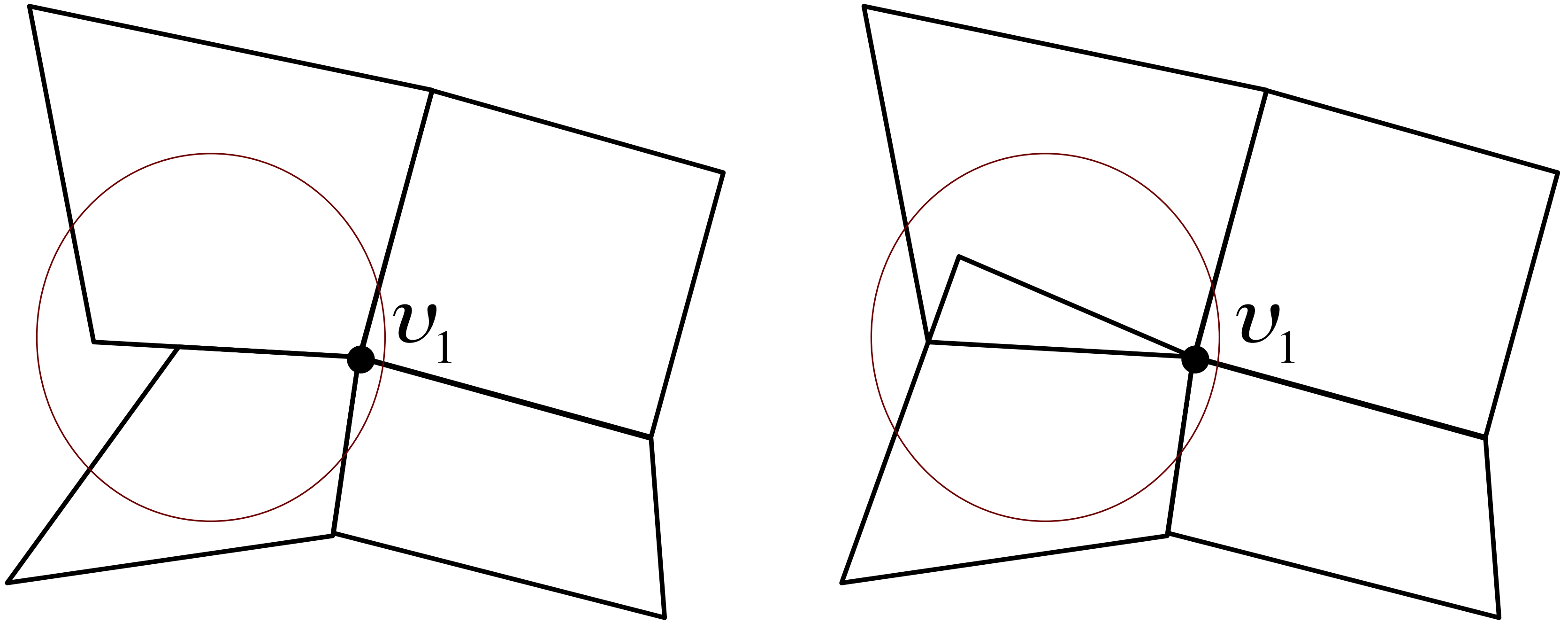


Consistency condition



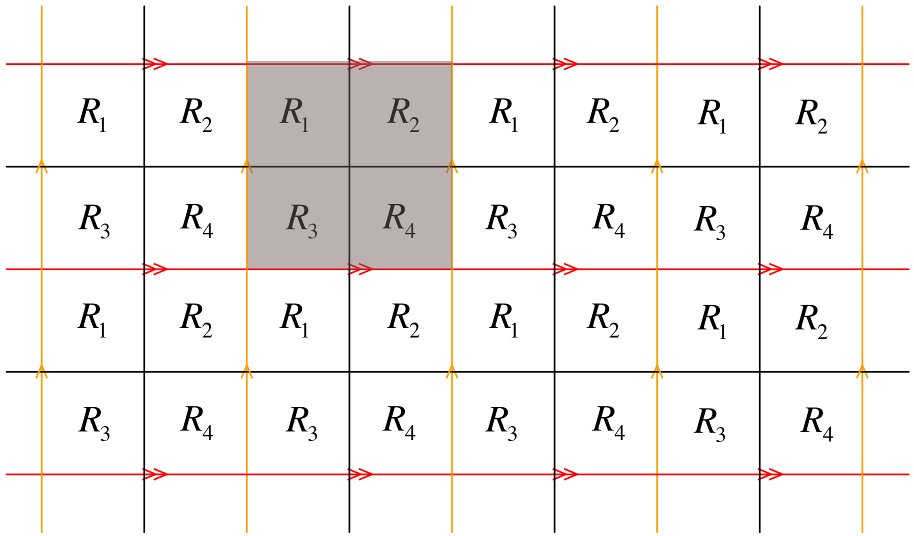


Consistency condition

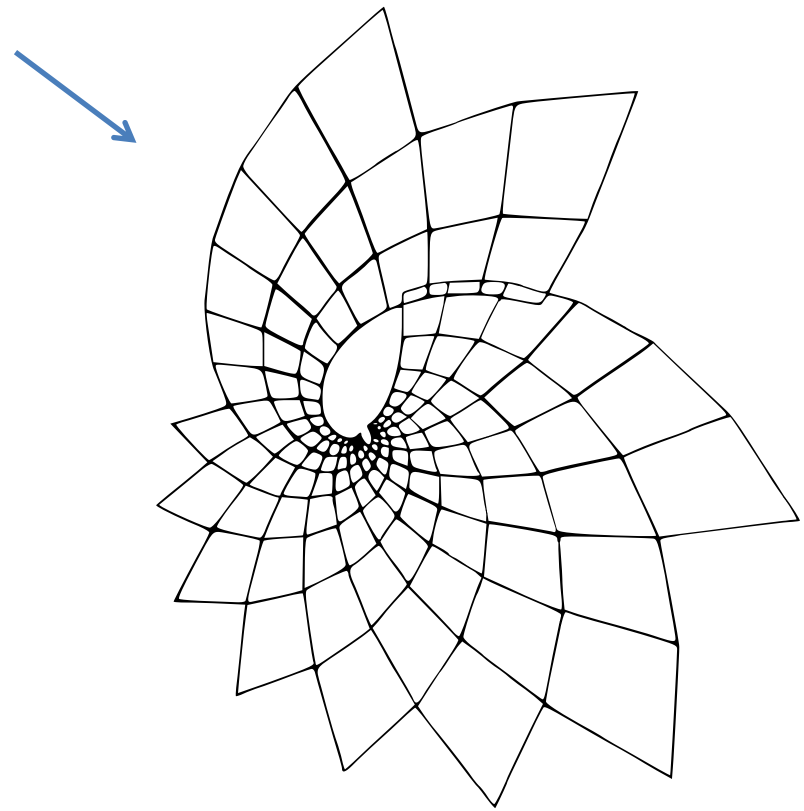


local similarity structure are not consistent around v_1

Developing map



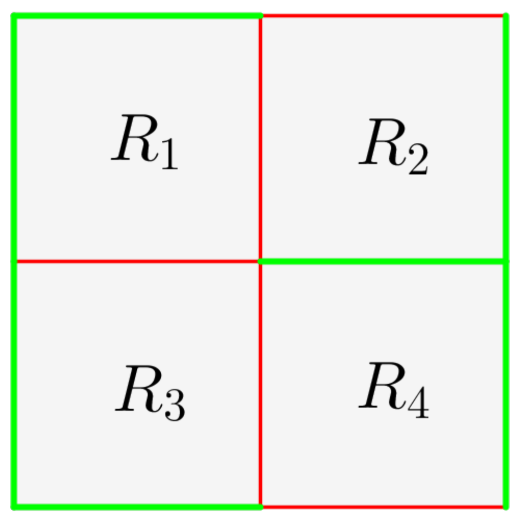
In general the image of a developing map is messy.



Problem

Find out sim. str. as above satisfying:

The image of the developing map with mount/valley assignment given by those in the following figure is a crease pattern s.t. each vertex satisfies the condition of (degree 4 flat folding) .



修士論文

2次元トーラスの相似構造による，一般化されたミウラ折りの構成

入井 美紀

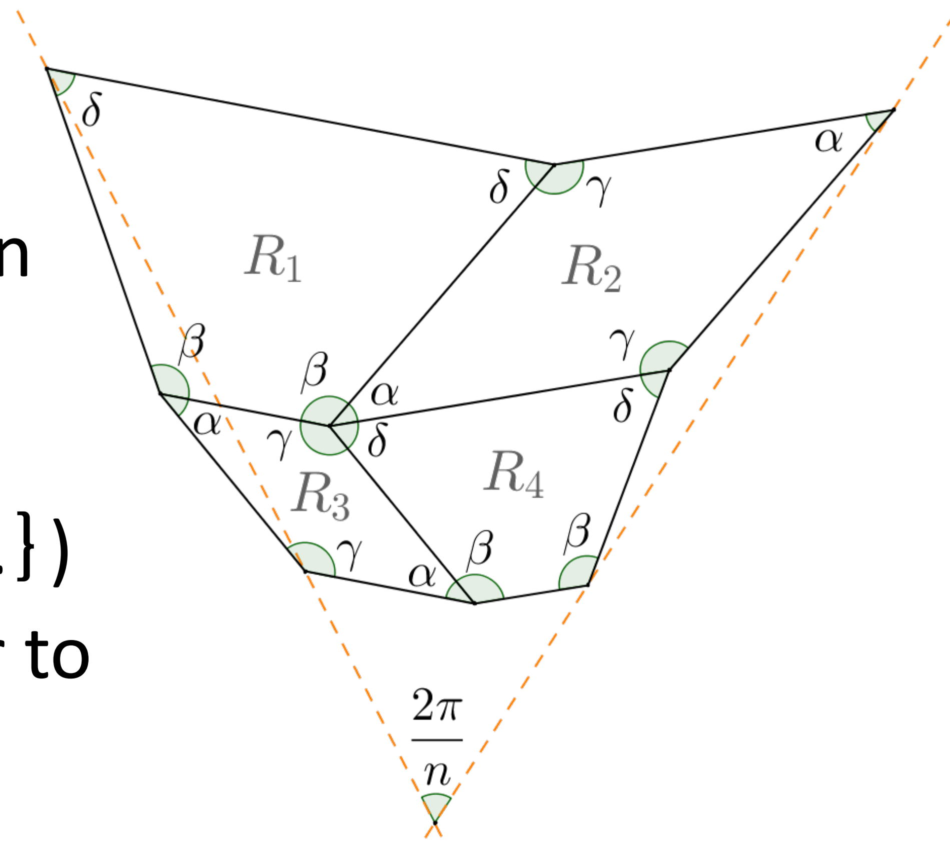
奈良女子大学大学院　人間文化研究科博士前期課程　数学専攻

2013年1月

The Result

Let R_1, R_2, R_3, R_4 be as:

- R_1, R_4 are similar trapezoids
- R_2, R_3 are parallelogram

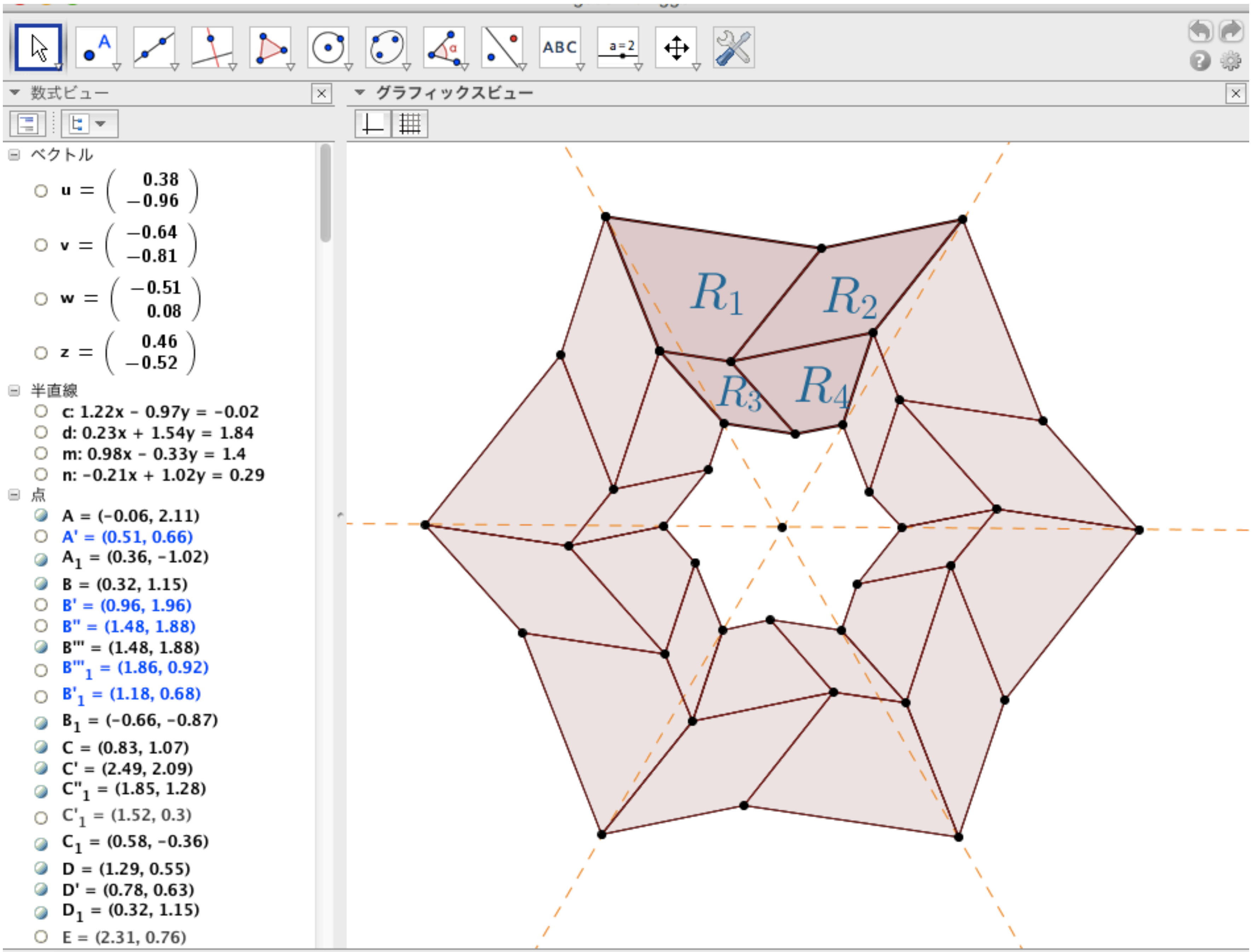


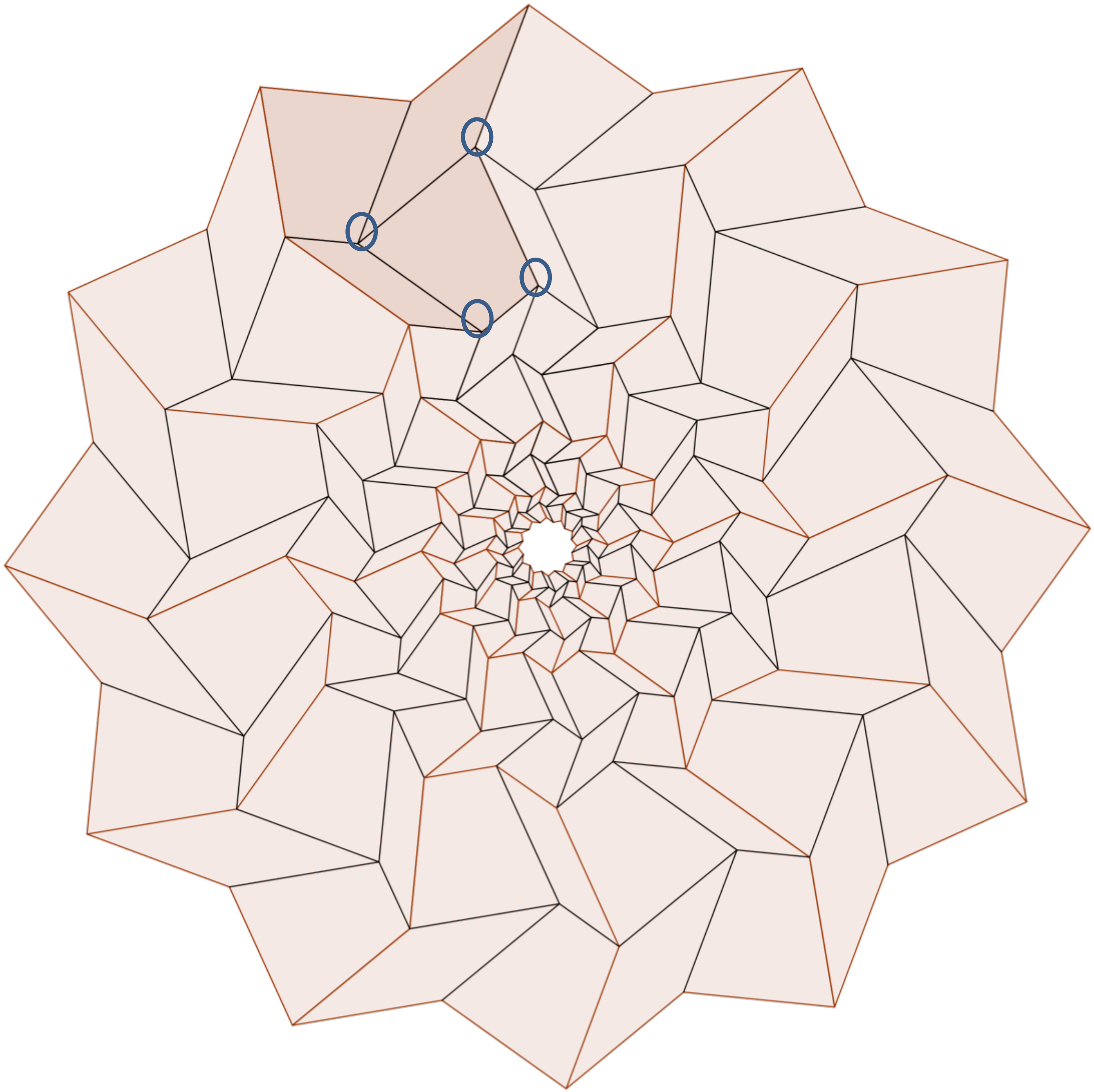
with angles α, β, γ, δ as in figure. Suppose:

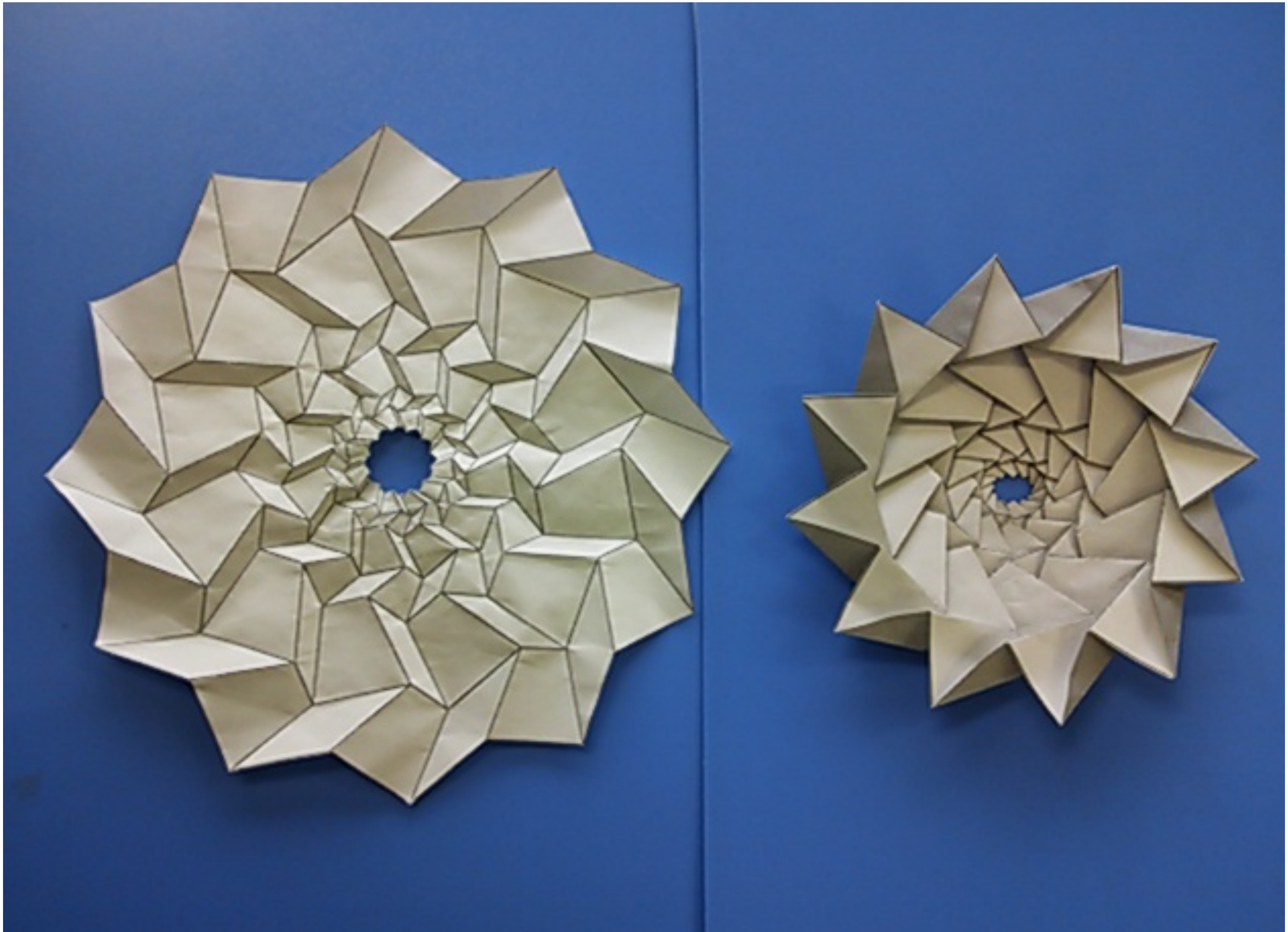
$\alpha < \beta < \gamma$, and

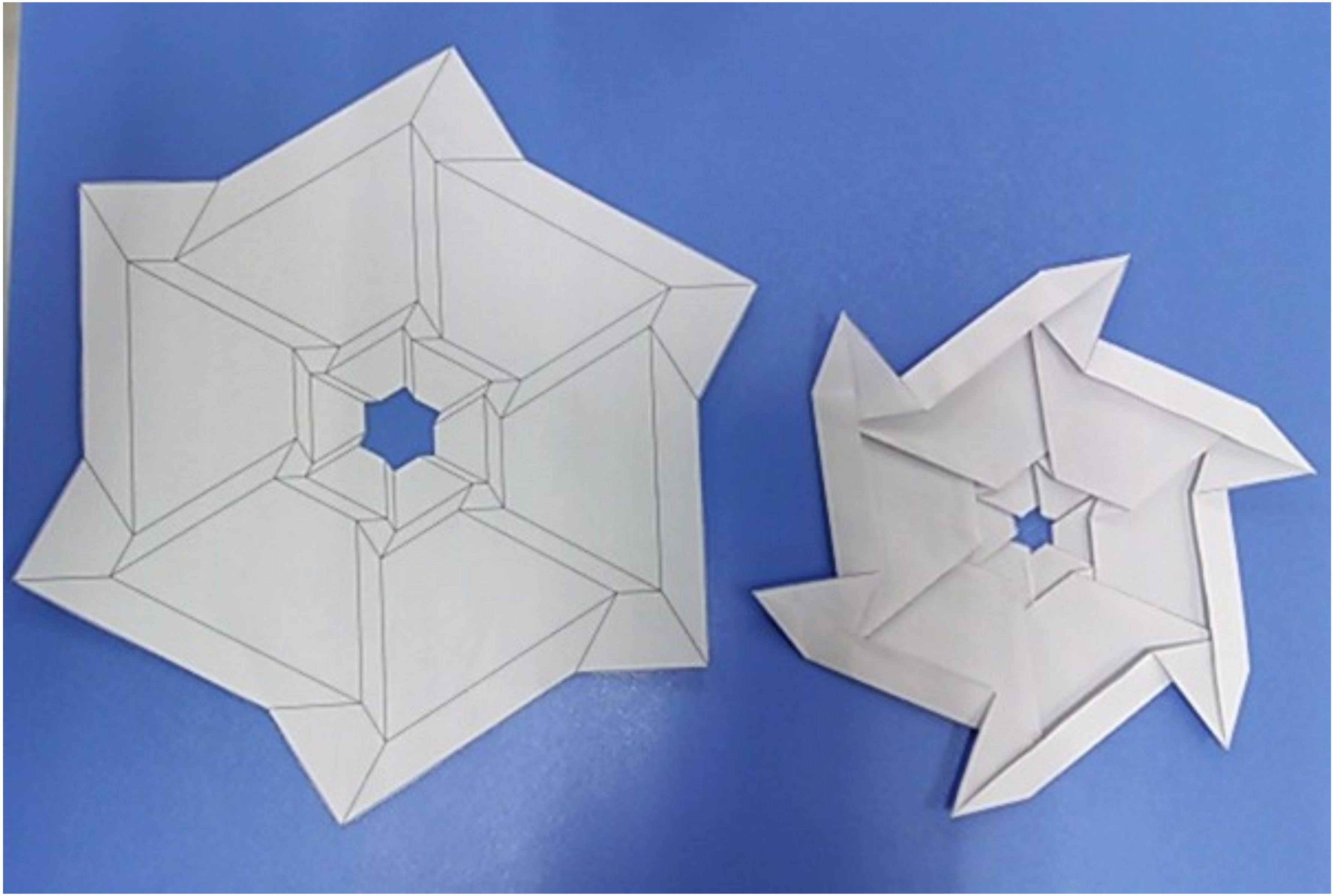
$\pi - 2\delta = 2\pi/n$ ($n \in \{3,4,\ldots\}$)

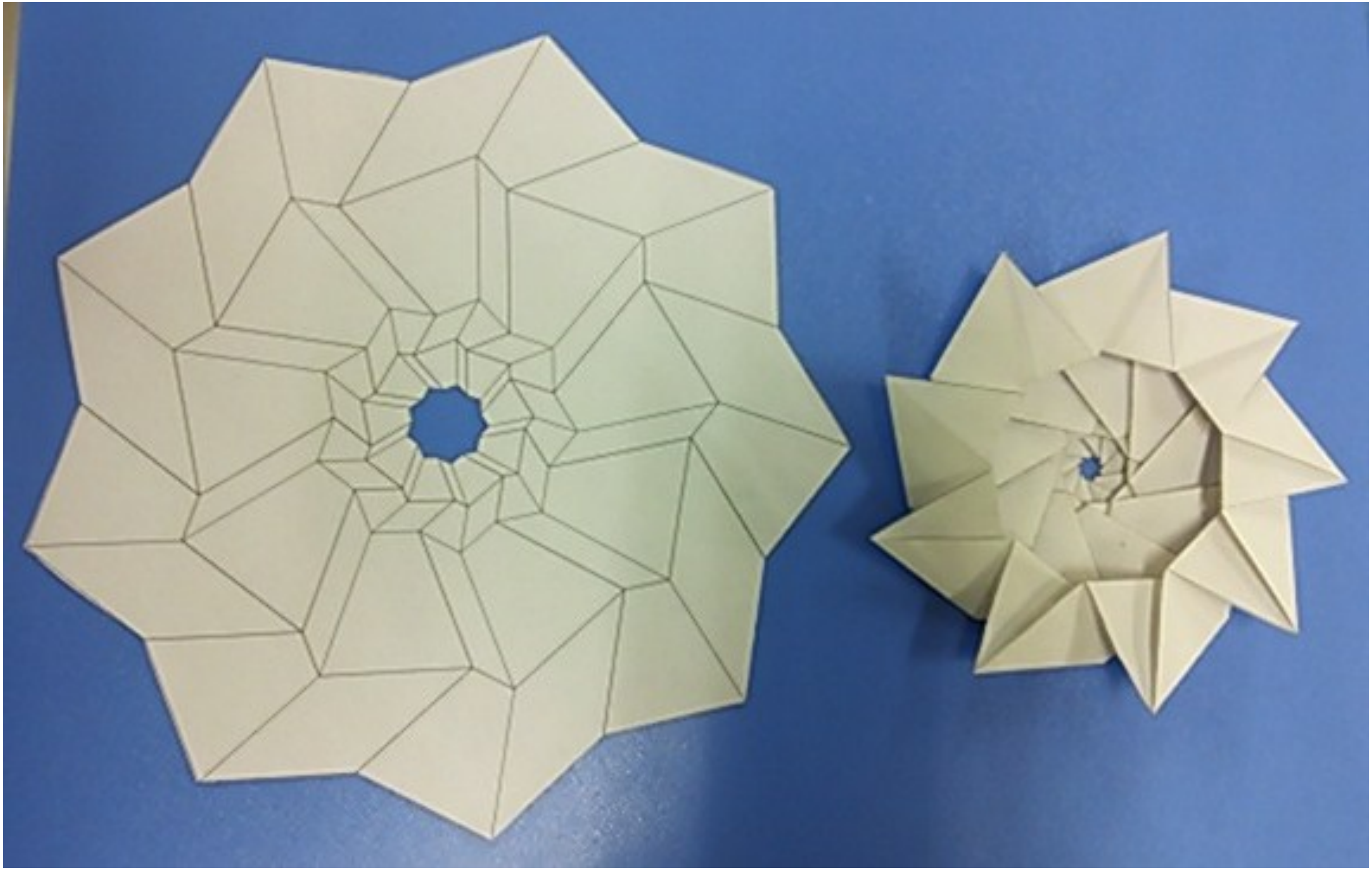
Then, this gives an answer to Problem.


ABC
a=2
数式ビュー
グラフィックスビュー
ベクトル
u = (0.38, −0.96)
v = (−0.64, −0.81)
w = (−0.51, 0.08)
z = (0.46, −0.52)
半直線
c: 1.22x − 0.97y = −0.02
d: 0.23x + 1.54y = 1.84
m: 0.98x − 0.33y = 1.4
n: −0.21x + 1.02y = 0.29
点
A = (−0.06, 2.11)
A' = (0.51, 0.66)
A1 = (0.36, −1.02)
B = (0.32, 1.15)
B' = (0.96, 1.96)
B'' = (1.48, 1.88)
B''' = (1.48, 1.88)
B'''1 = (1.86, 0.92)
B'1 = (1.18, 0.68)
B1 = (−0.66, −0.87)
C = (0.83, 1.07)
C' = (2.49, 2.09)
C''1 = (1.85, 1.28)
C'1 = (1.52, 0.3)
C1 = (0.58, −0.36)
D = (1.29, 0.55)
D' = (0.78, 0.63)
D1 = (0.32, 1.15)
E = (2.31, 0.76)
R1
R2
R3
R4







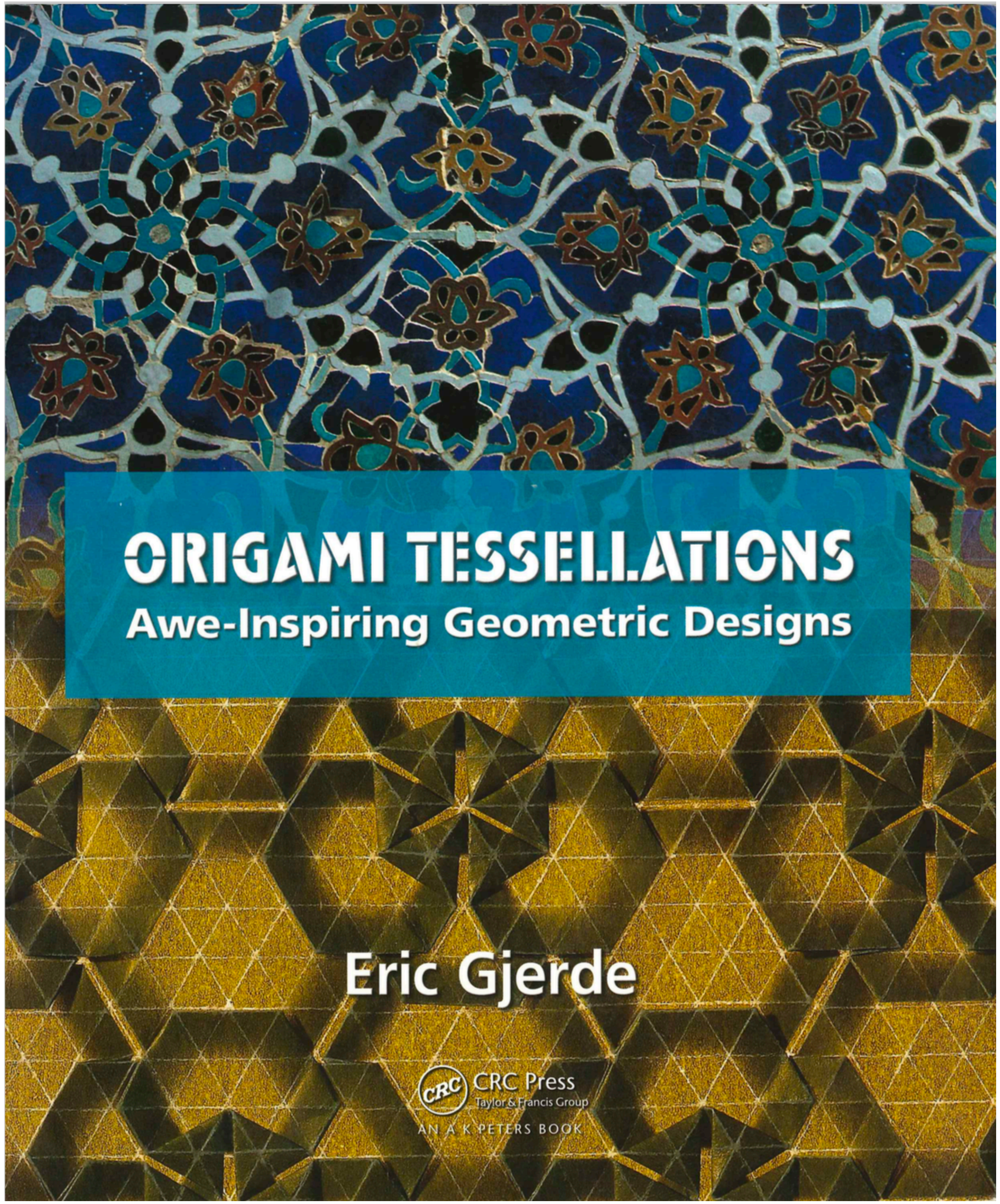


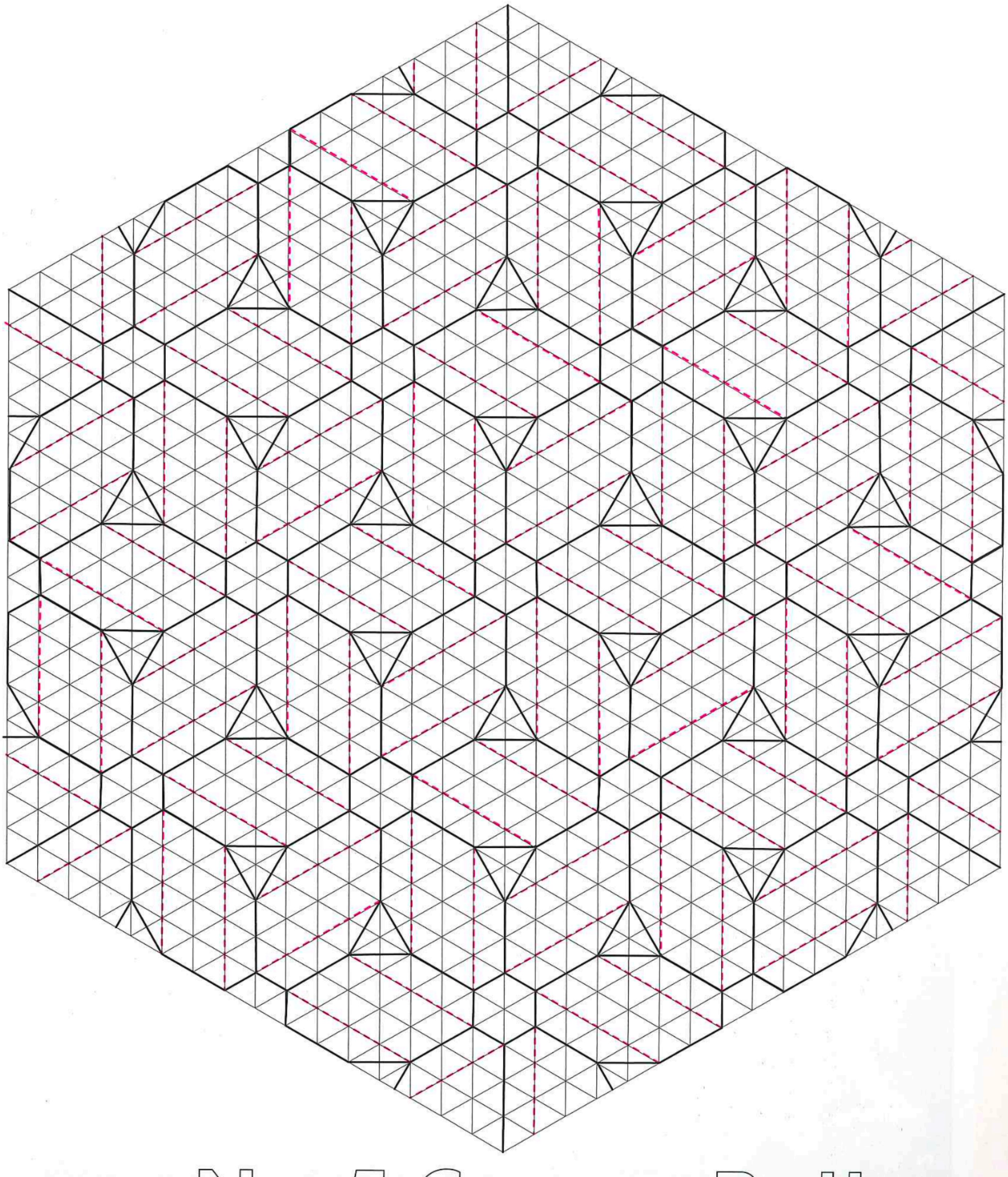
Further research

2-dimensional Euclidean orbifold:

THE GEOMETRIES OF 3-MANIFOLDS

PETER SCOTT


ORIGAMI TESSELLATIONS
Awe-Inspiring Geometric Designs
Eric Gjerde
CRC Press
Taylor & Francis Group
AN A K PETERS BOOK



No.5 Crease Pattern

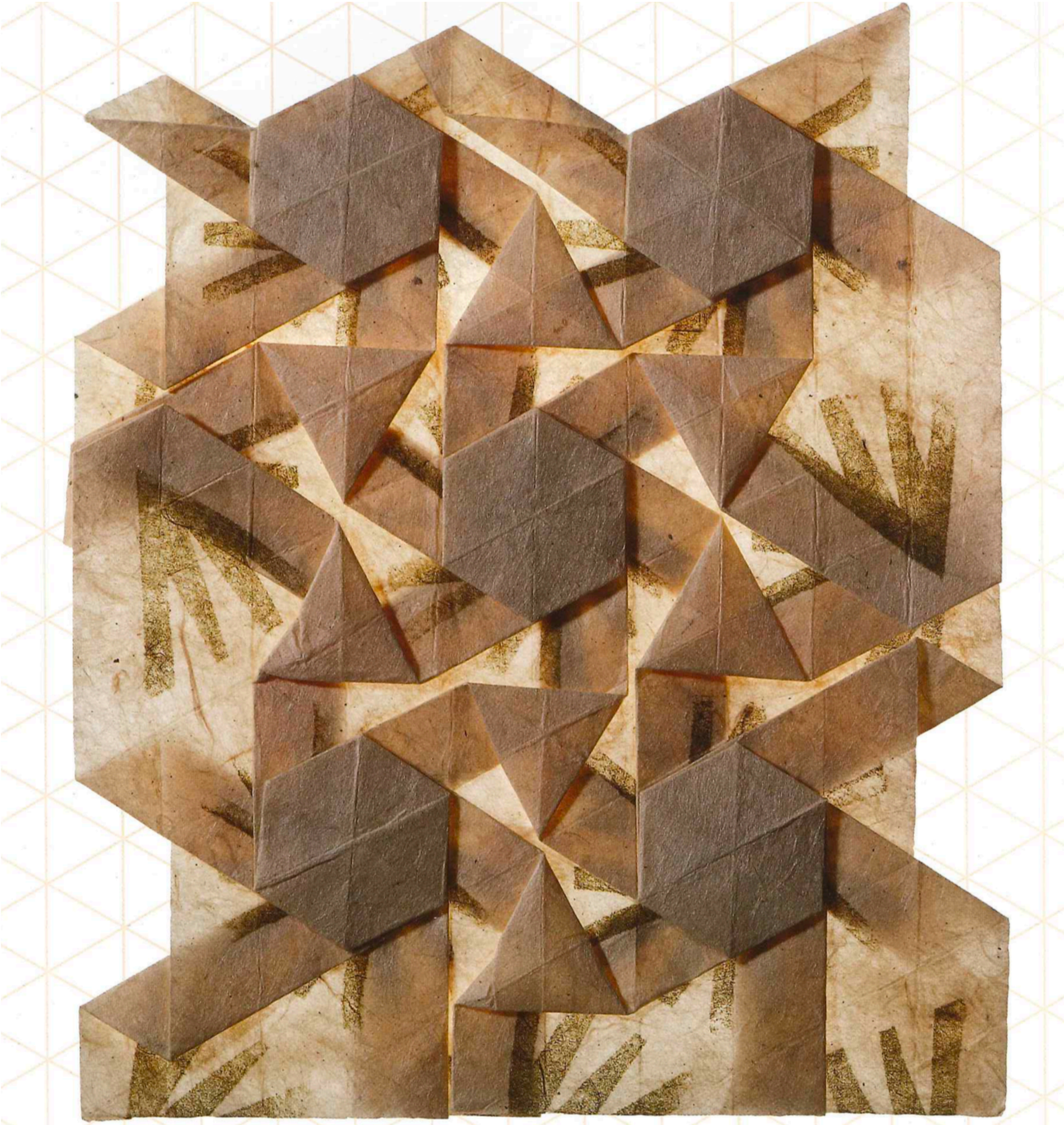


TABLE 4.4

THE SEVENTEEN CLOSED 2-DIMENSIONAL EUCLIDEAN ORBIFOLDS

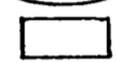
The integers in brackets specify the cone angles. Thus (n) denotes a cone angle $2\pi/n$. Note that all boundary curves are reflector curves.

Underlying surface X	*Cone points*	*Number of Seifert bundles over X with e = 0*	*Number with orientable total space*
Torus		3	1
Klein bottle		5	1
S^2	(2, 2, 2, 2)	1	1
S^2	(2, 4, 4)	1	1
S^2	(2, 3, 6)	1	1
S^2	(3, 3, 3)	1	1
P^2	(2, 2)	2	1
Annulus		2	0
Moebius band		2	0
D^2	(2, 2)	1	0

これらについて存在するかどうか調べる

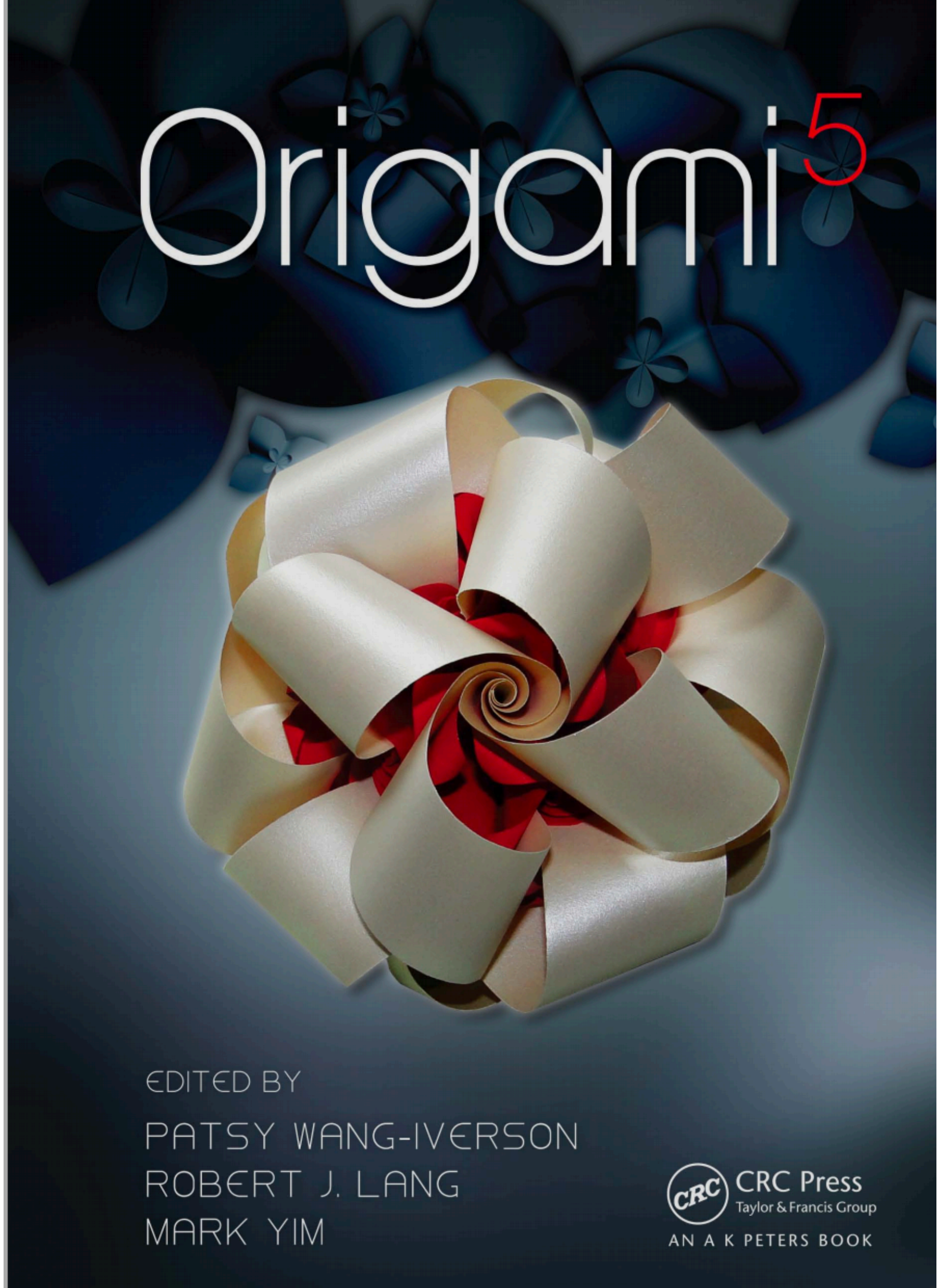
The remaining seven orbifolds all have corner reflectors and have underlying surface D^2.

In the first column, an integer (n) specifies a cone angle of $2\pi/n$. In the second column, an integer (n) specifies an angle of π/n.

Cone points	*Corner reflectors*	*Picture*
(2)	(2, 2)	
(3)	(3)	
(4)	(2)	
	(2, 2, 2, 2)	
	(2, 4, 4)	
	(2, 3, 6)	
	(3, 3, 3)	

存在するかどうか調べる

Second construction:
Shrink and Rotate


Origami5
EDITED BY
PATSY WANG-IVERSON
ROBERT J. LANG
MARK YIM
CRC Press
Taylor & Francis Group
AN A K PETERS BOOK

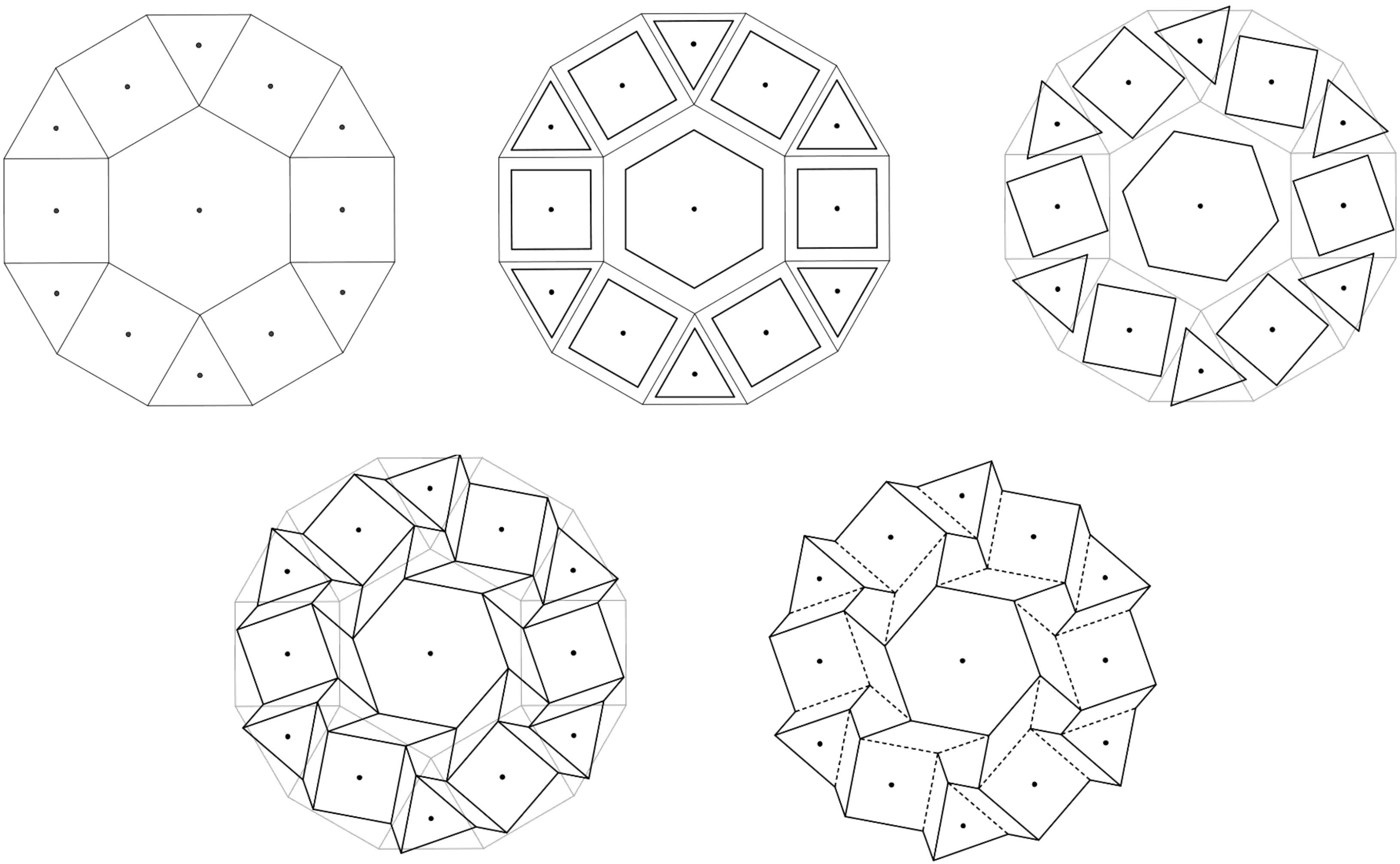
Shrink and Rotate

Every Spider Web Has a
Simple Flat Twist Tessellation

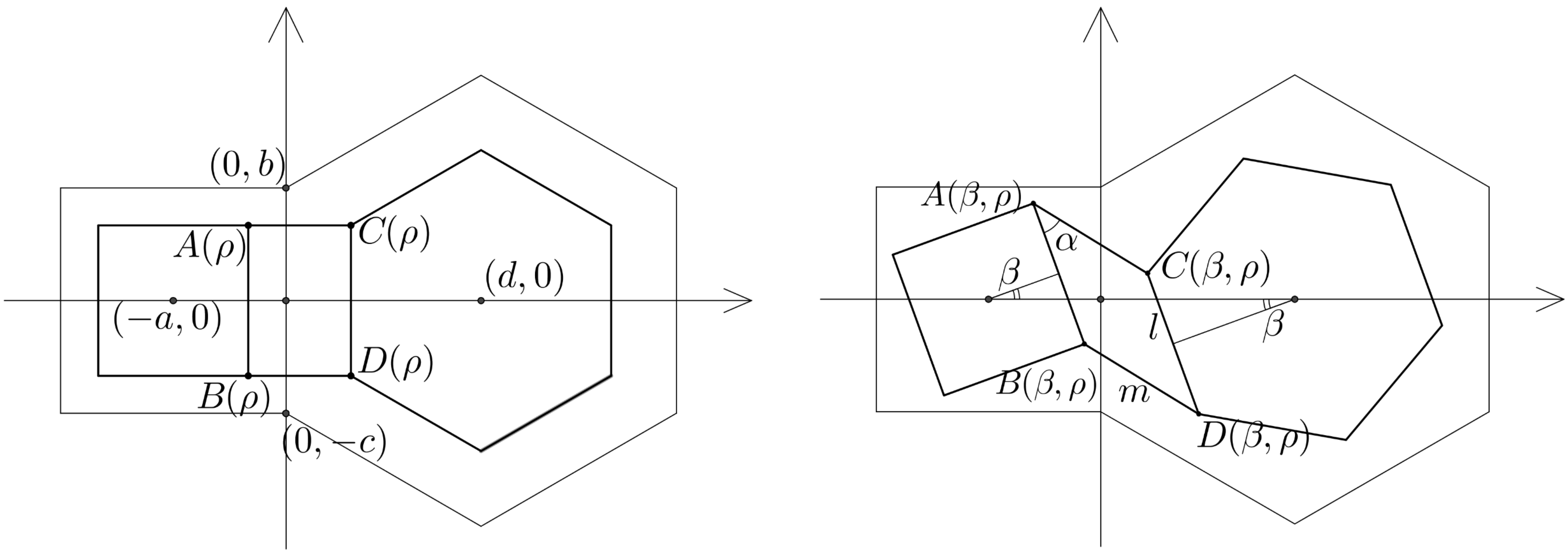
Robert J. Lang and Alex Bateman

In this paper, Lang and Bateman make detailed analysis of the construction of flat fold. Origami, called Shrink and Rotate, proposed by Bateman

Shrink and Rotate



Why “Shrink and Rotate” works?

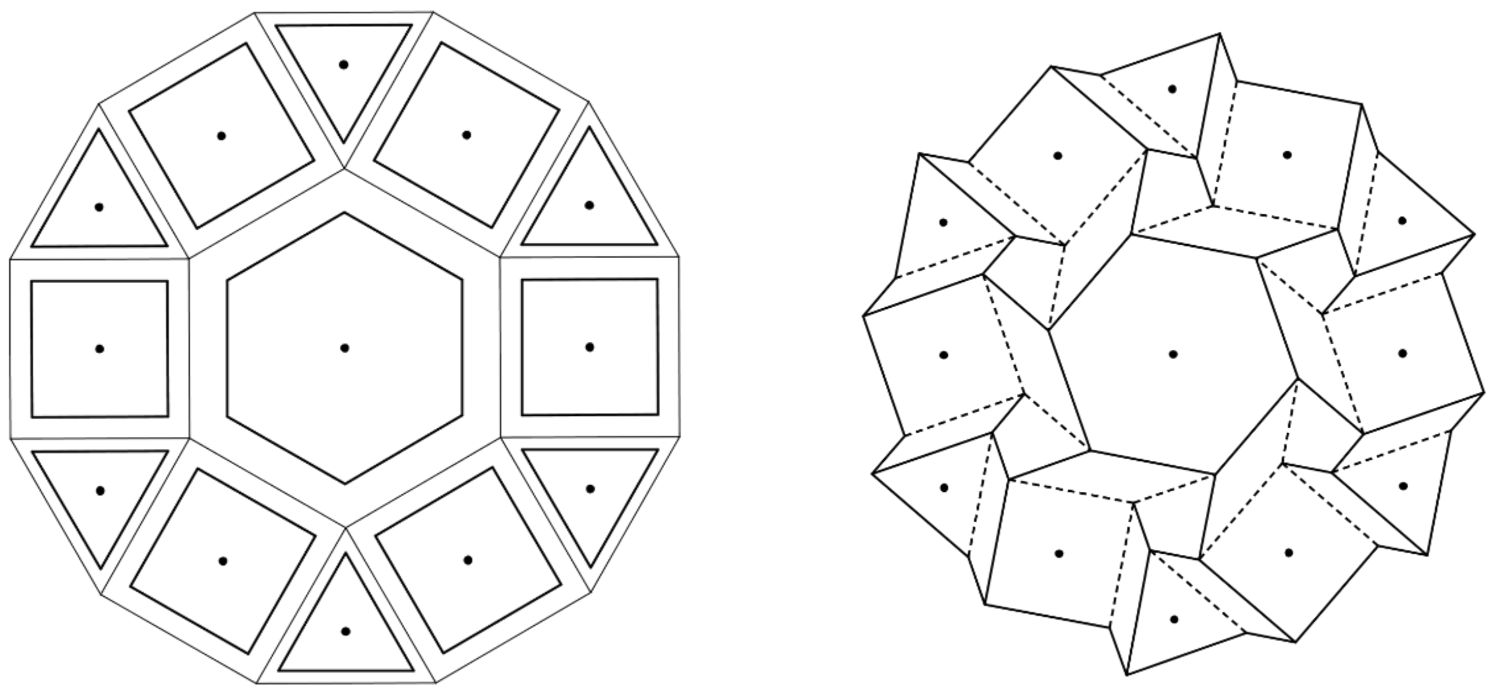


Previous high school math can show:

$$\alpha = \cos^{-1}\left(\frac{\sin\beta}{\sqrt{\rho^2 - 2\rho\cos\beta + 1}}\right)$$

Note that α depends β and ρ only.

This shows :
each vertex of the obtained crease pattern satisfies the flat foldability condition



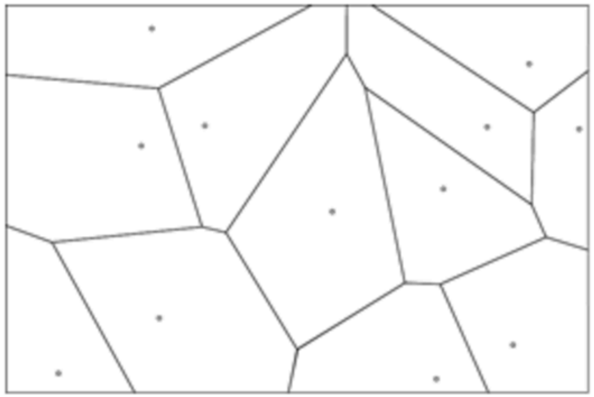
You might think that the argument works for more general tessellations.

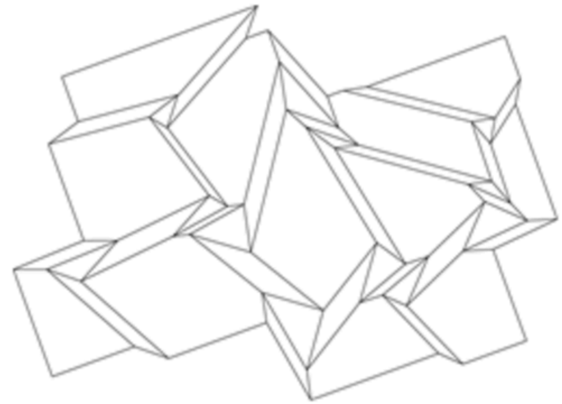
Yes, that is true.

In fact, the argument works for

Voronoi tessellation.

Voronoi origami





詳しくは三谷先生のウェブページを御覧ください。
http://junmitani.hatenablog.com/entry/20130516

Crease pattern and Folded origami

I and a student of mine, Atsumi Hokkyo, examined the paper and encountered:

Consider the following: if we construct two shrink-rotate tessellations from a given tiling using a twist angle α for one of them and $-\alpha$ for the other, then all of the polygons in the former will be similar (in the strict geometric sense) to the corresponding polygon in the other, with a single scaling constant between every pair of polygons. What's more, in the corresponding parallelograms, the angles that become twist angles are equal and opposite—which means the same relation must be true for the opposite

Theorem 1 (Crease Pattern/Folded Form Duality). *If a shrink-rotate tessellation crease pattern is constructed from a tiling using aspect ratio parameter and twist angle* (γ, α)*, then the folded form of that tessellation is given to within a scaling constant by the same construction using* $(\gamma, -\alpha)$*.*

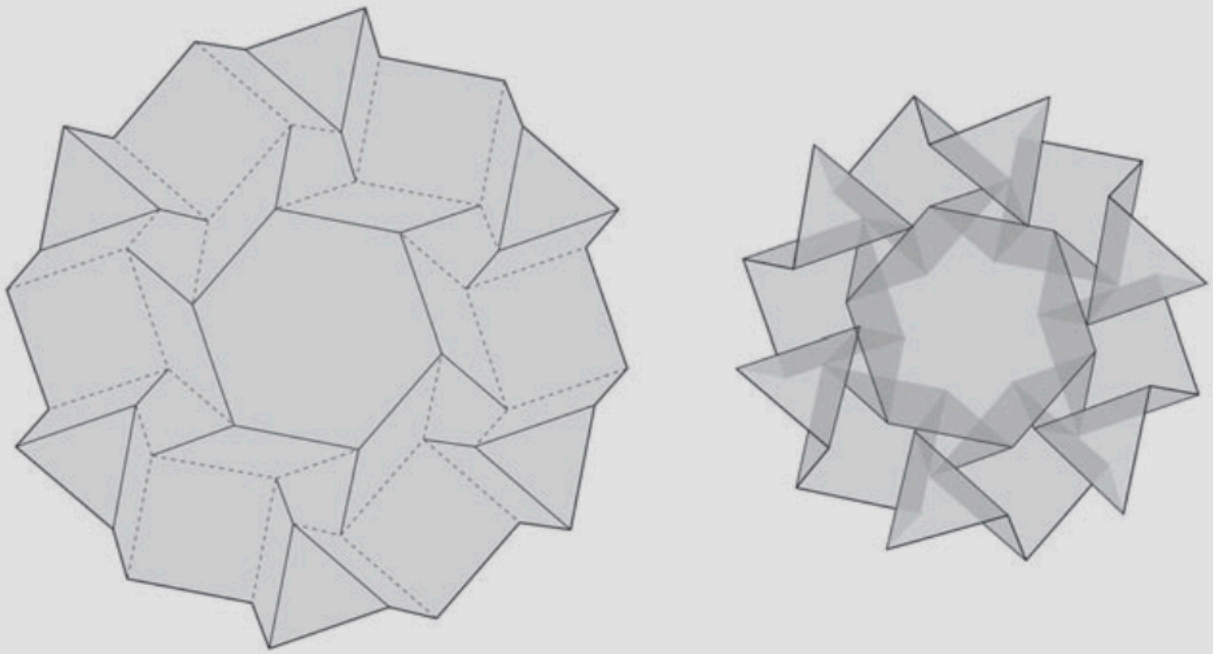


Figure 4. The crease pattern (left), and the folded form (right).

In her master thesis, Hokkyo showed that the statement does not hold in general.

ここで，$-\varepsilon < \theta < \varepsilon$ で $f(\theta)$ は単調増加だから，$f(\theta_1) > f(\theta_2)$ となる．よって，ねじり角 α の平行四辺形とねじり角 $-\alpha$ の平行四辺形は合同にならない．

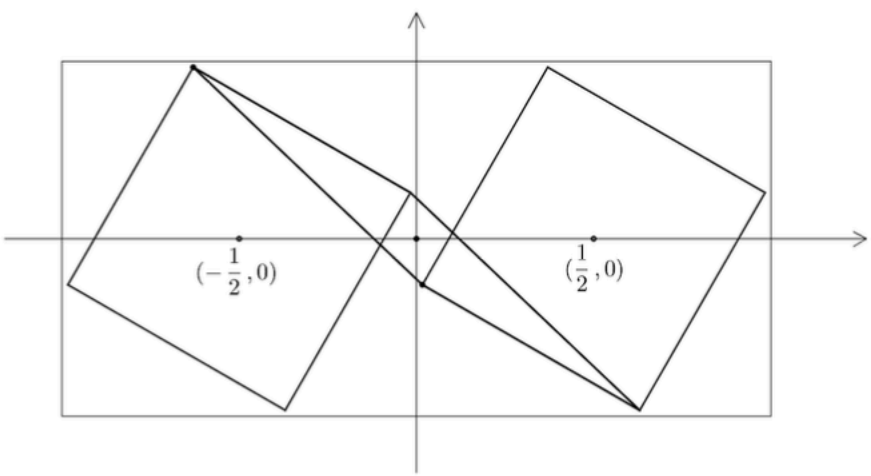


図 4.11　θ 回転

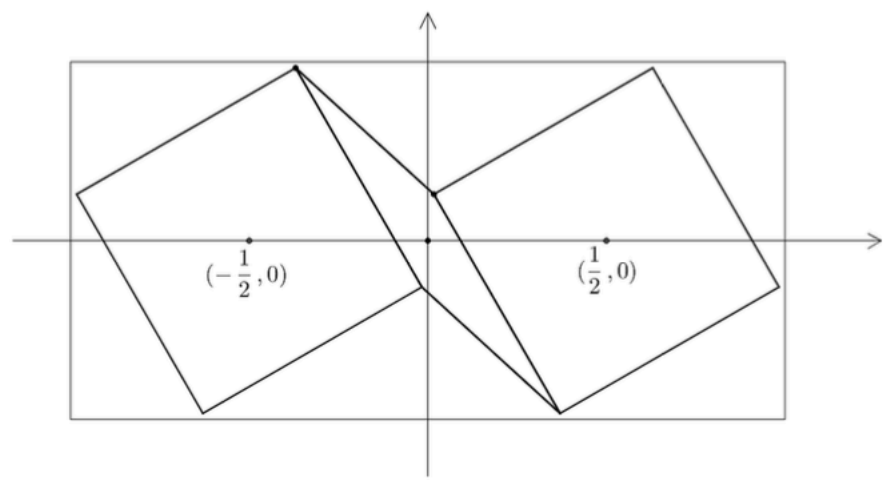


図 4.12　$-\theta$ 回転

Third construction:
Hokkyo’s construction

Hokkyo’s construction

修士論文

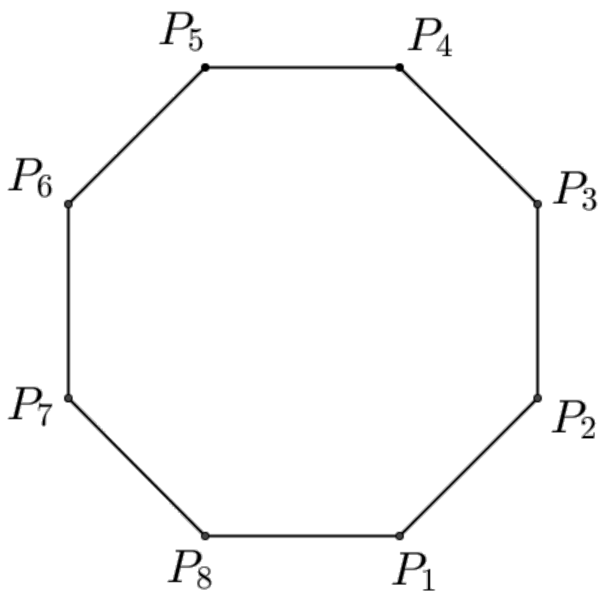
Origami Tessellation の新しい構成方法について

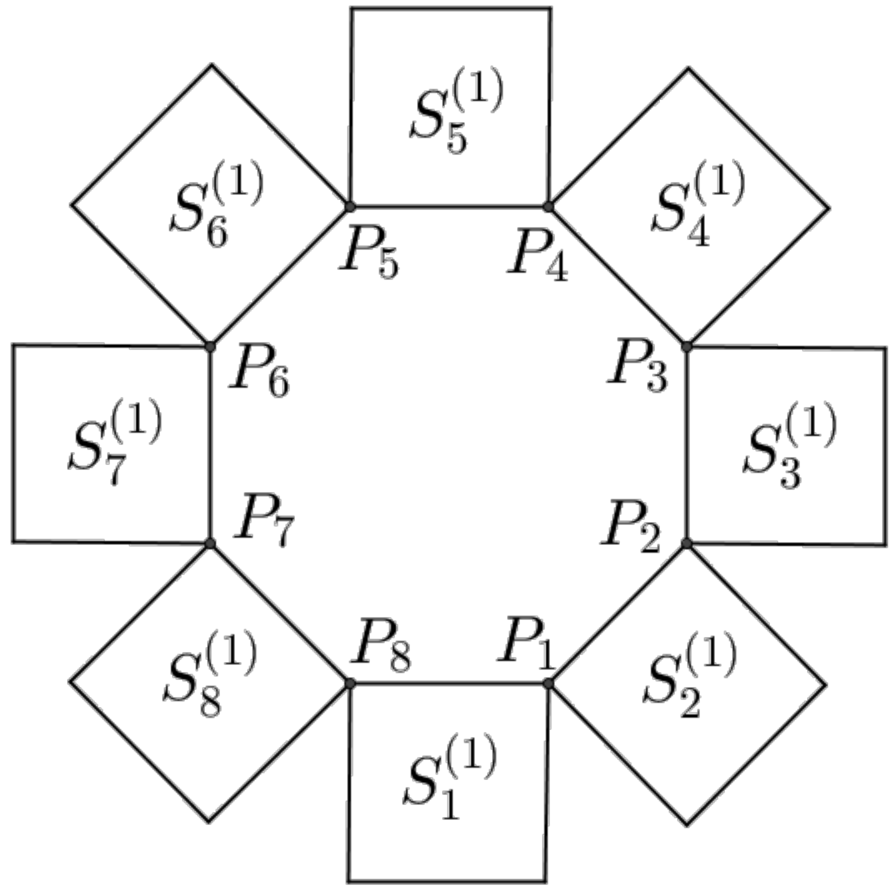
法橋　厚美

奈良女子大学大学院　人間文化研究科博士前期課程　数学専攻

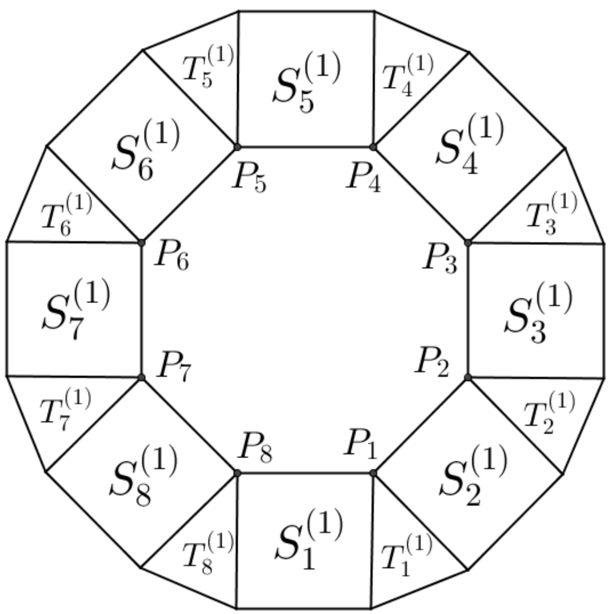
2016 年 1 月

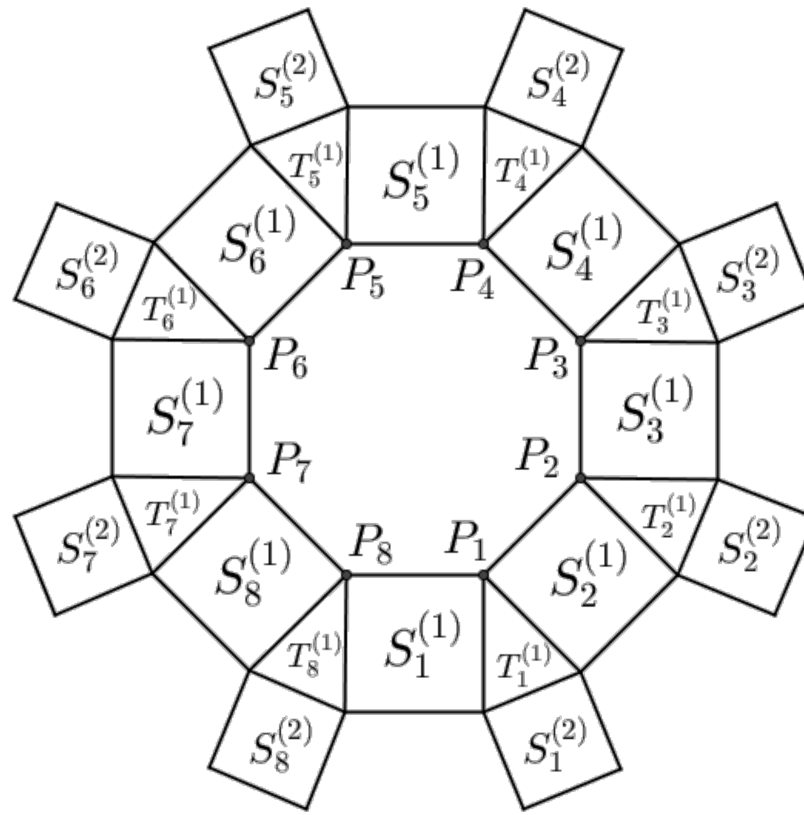
Hokkyo's construction



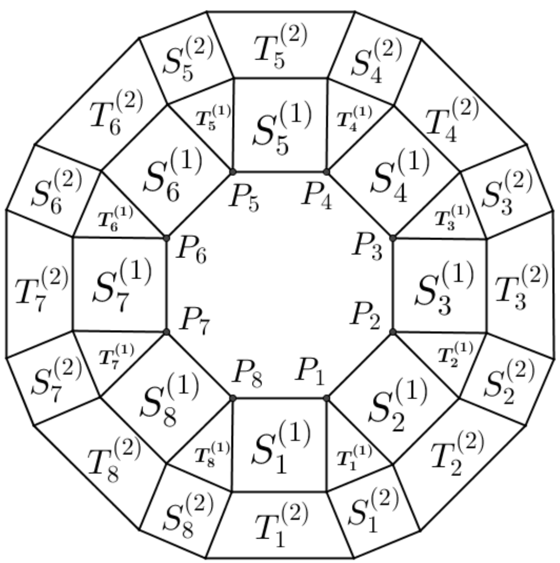


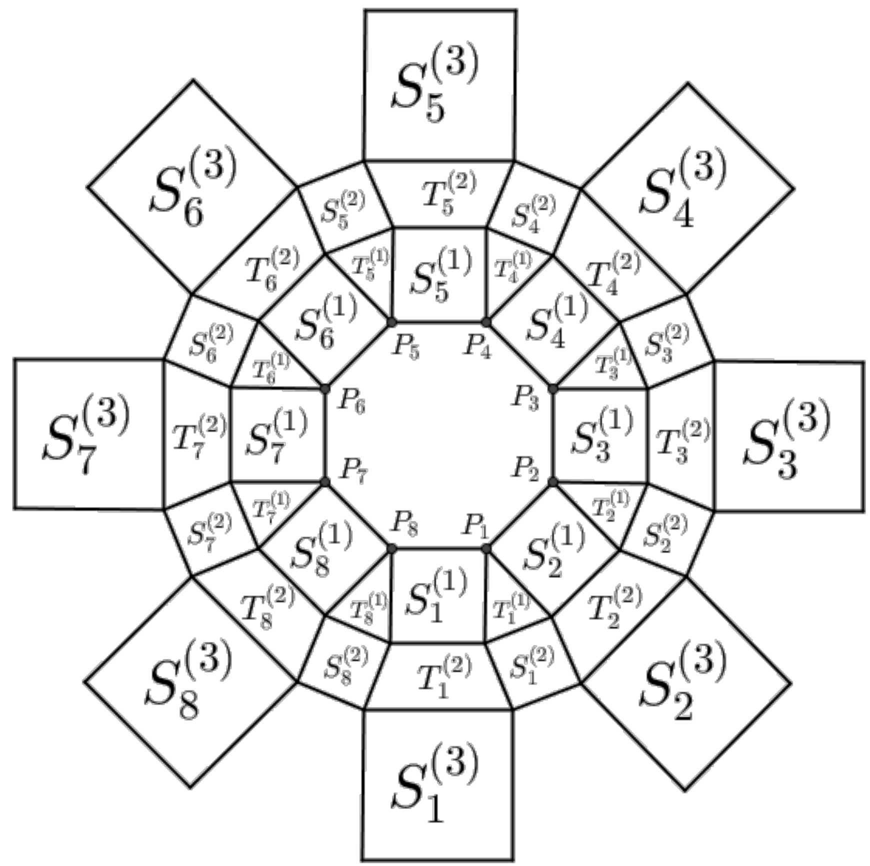
Hokkyo’s construction



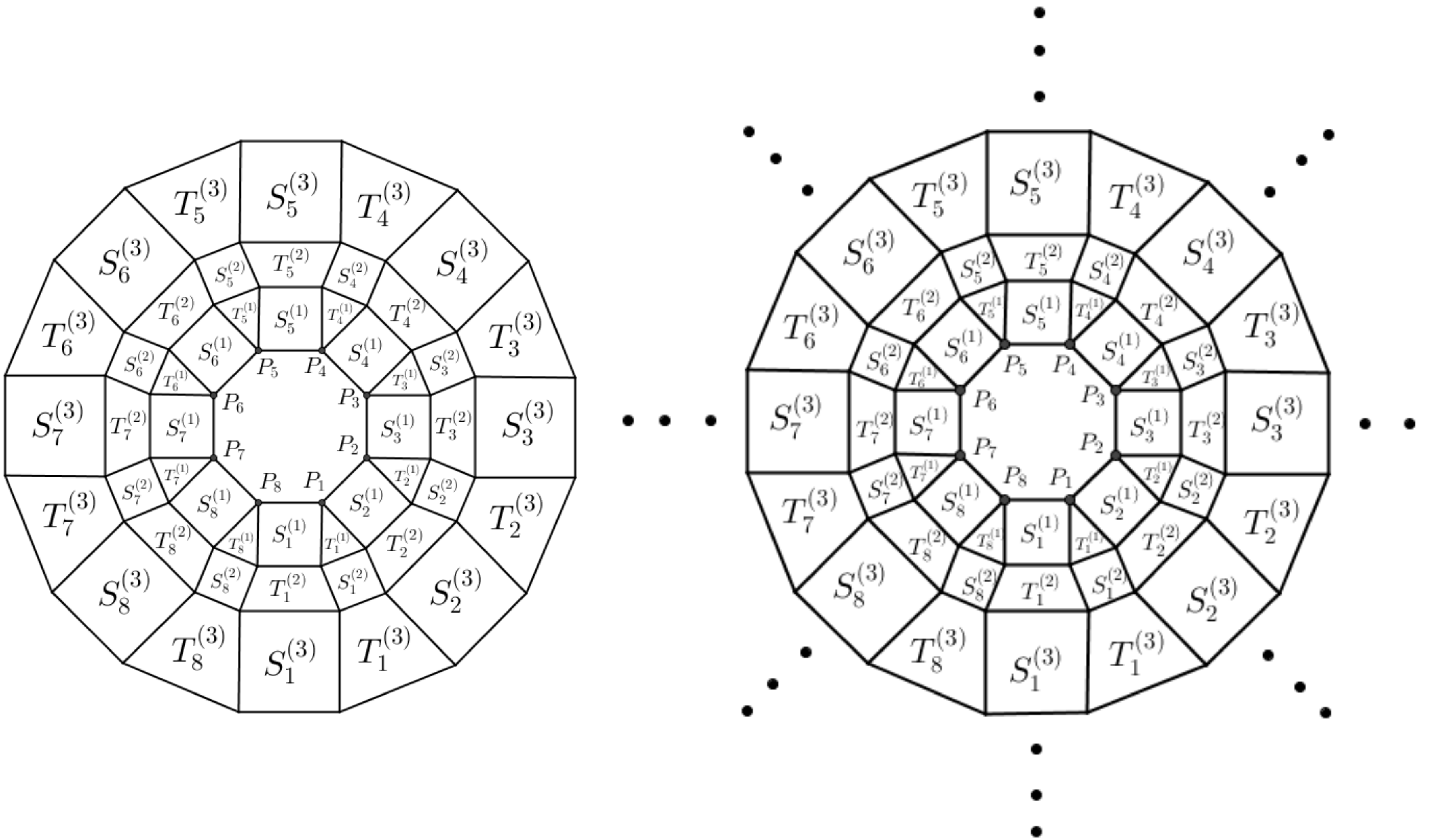


Hokkyo’s construction

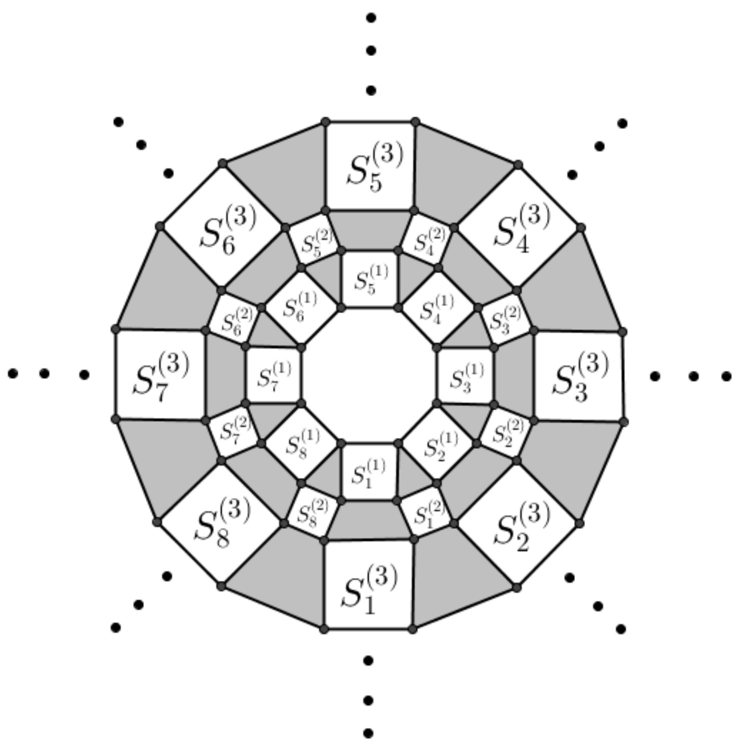


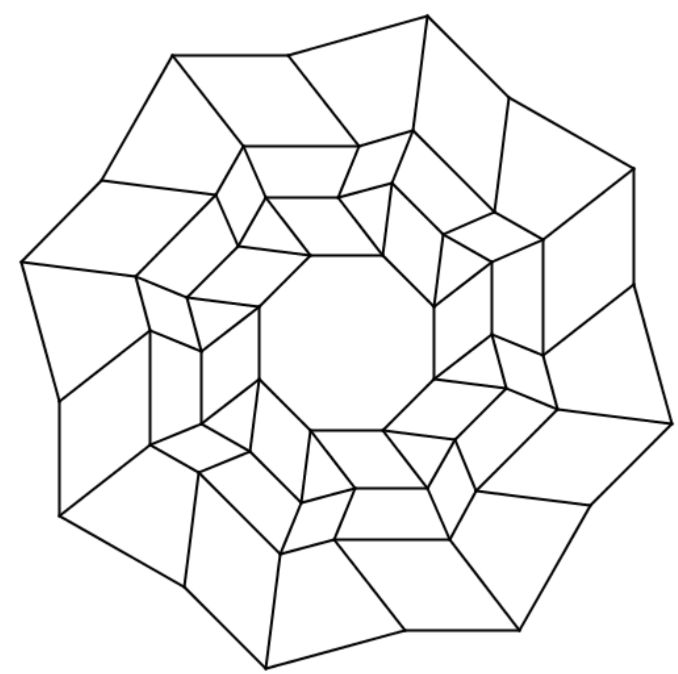


Hokkyo's construction

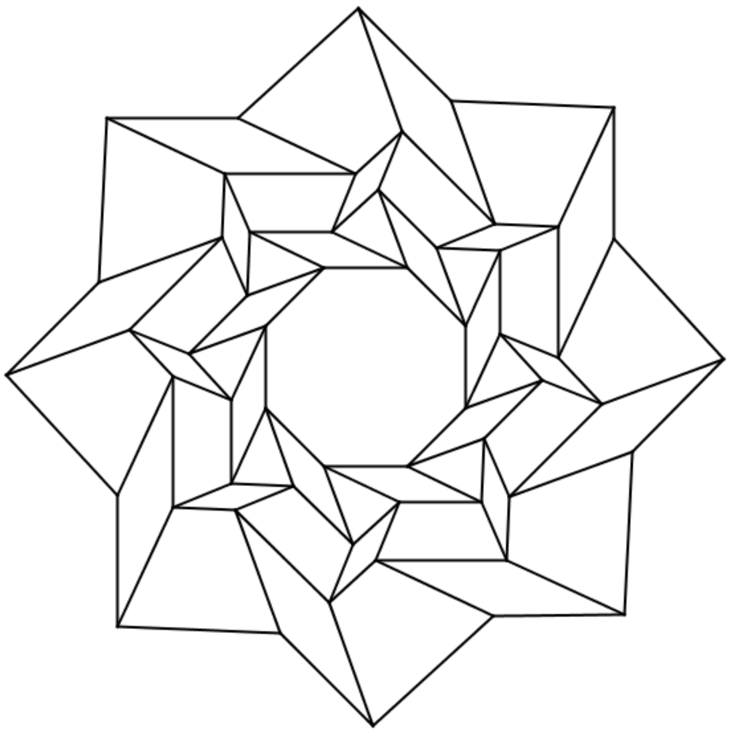


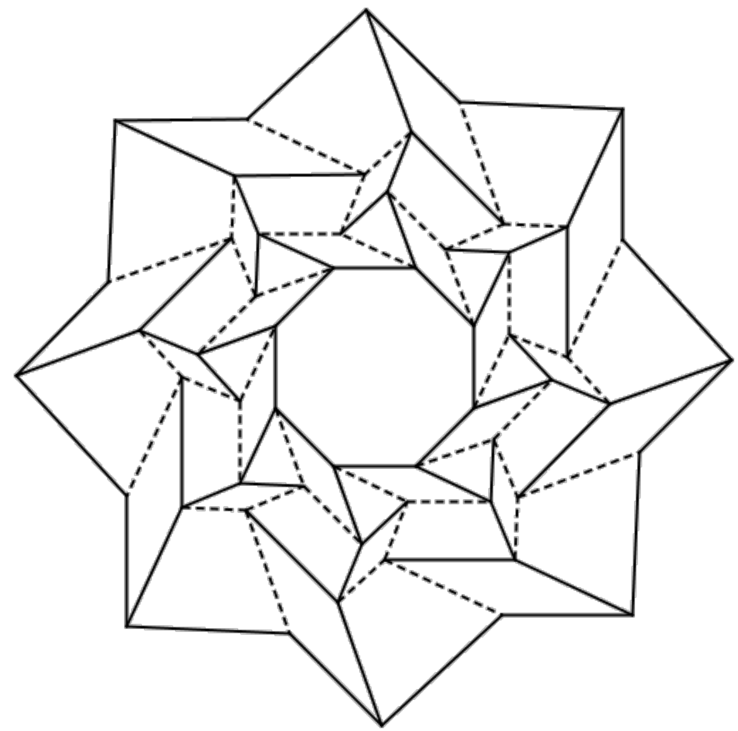
Hokkyo's construction





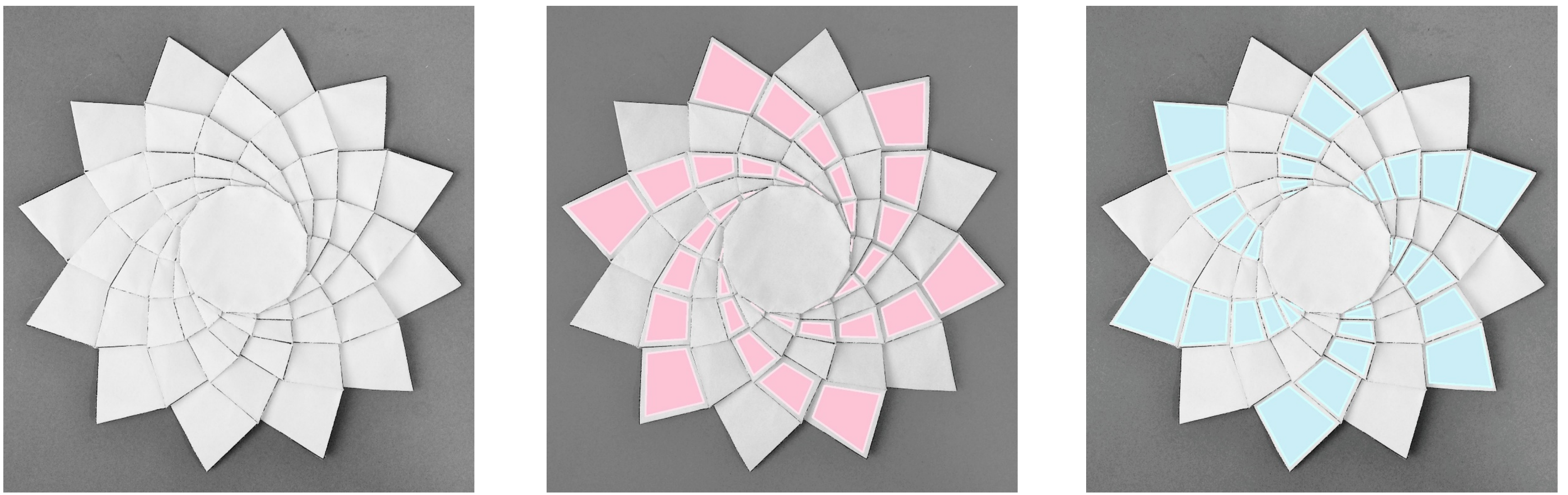
Hokkyo’s construction

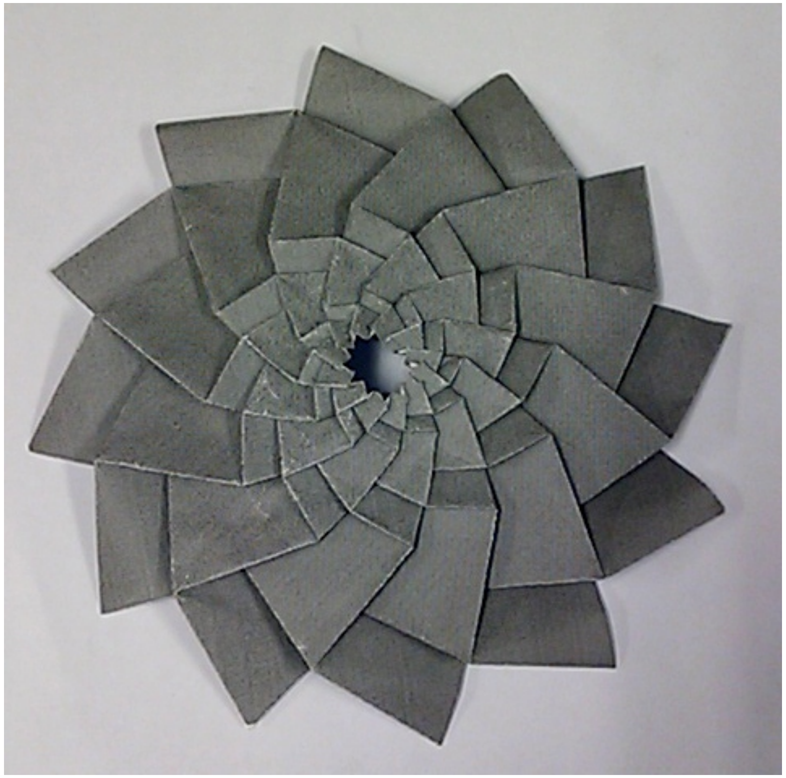




Hokkyo’s construction

We can observe spirals in two directions.



Interesting Feature

The statement of Theorem 1(Lang-Bateman) holds for Hokkyo's origami.

Question

Is there a “mathematical structure” in Hokkyo’s origami ?