On the 4-dimensional smooth Smale conjecture

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Diff(M) = group of diffeomorphisms $M \to M$ with the C^∞-topology

Diff(S^1) $\simeq O(2)$ (easy)
Diff(S^2) $\simeq O(3)$ (Smale, 1959)
Diff(S^3) $\simeq O(4)$ (Smale conjecture, proved by A. Hatcher, 1983)
Diff(S^d) $\not\simeq O(d+1)$ for $d \geq 5$ (Novikov, Antonelli-Burghelea-Kahn,...)

4-dim Smale conjecture
(Problem 4.34, 4.126 in [Kirby])

$$\mathrm{Diff}(S^4) \simeq O(5) \qquad (*)$$

Rem. $\mathrm{Diff}(S^d) \simeq O(d+1) \times \mathrm{Diff}(D^d, \partial)$ holds.
Hence, $(*) \Leftrightarrow \mathrm{Diff}(D^4, \partial) \simeq \mathrm{pt}$. id on a nh of ∂

Theorem (W) $\mathrm{Diff}(D^4, \partial) \not\simeq \mathrm{pt}$.
Hence, $(*)$ is false.

Kontsevich's map

Kontsevich's characteristic class gives a hom

$$I : H(\mathcal{G}) \to H^*(B\mathrm{Diff}(D^4, \partial); \mathbf{R})$$

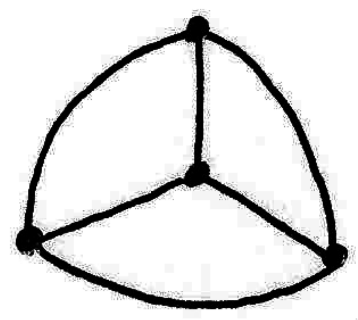
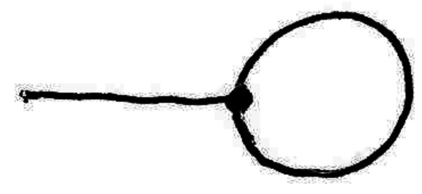
($H(\mathcal{G})$: graph homology)

Theorem (W) *Kontsevich map is injective on trivalent part $H_{\mathrm{tri}}(\mathcal{G}) \subset H(\mathcal{G})$.*

Rem. If $\mathrm{Diff}(D^4, \partial) \simeq \mathrm{pt}$, then
$H^*(B\mathrm{Diff}(D^4, \partial); \mathbf{R}) = H^*(\mathrm{pt}; \mathbf{R})$
Thus, it suffices to check $H_{\mathrm{tri}}(\mathcal{G}) \neq 0$.

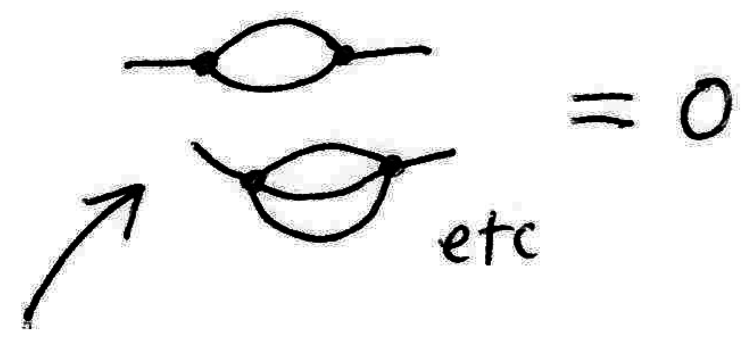
Problem Compute $H_{\mathrm{tri}}(\mathcal{G})$.

1. Graph homology

graph: $\Gamma =$  etc. without 

(valence≥3)

orientation: ori. of $\mathbf{R}^{\mathrm{Edges}(\Gamma)}$



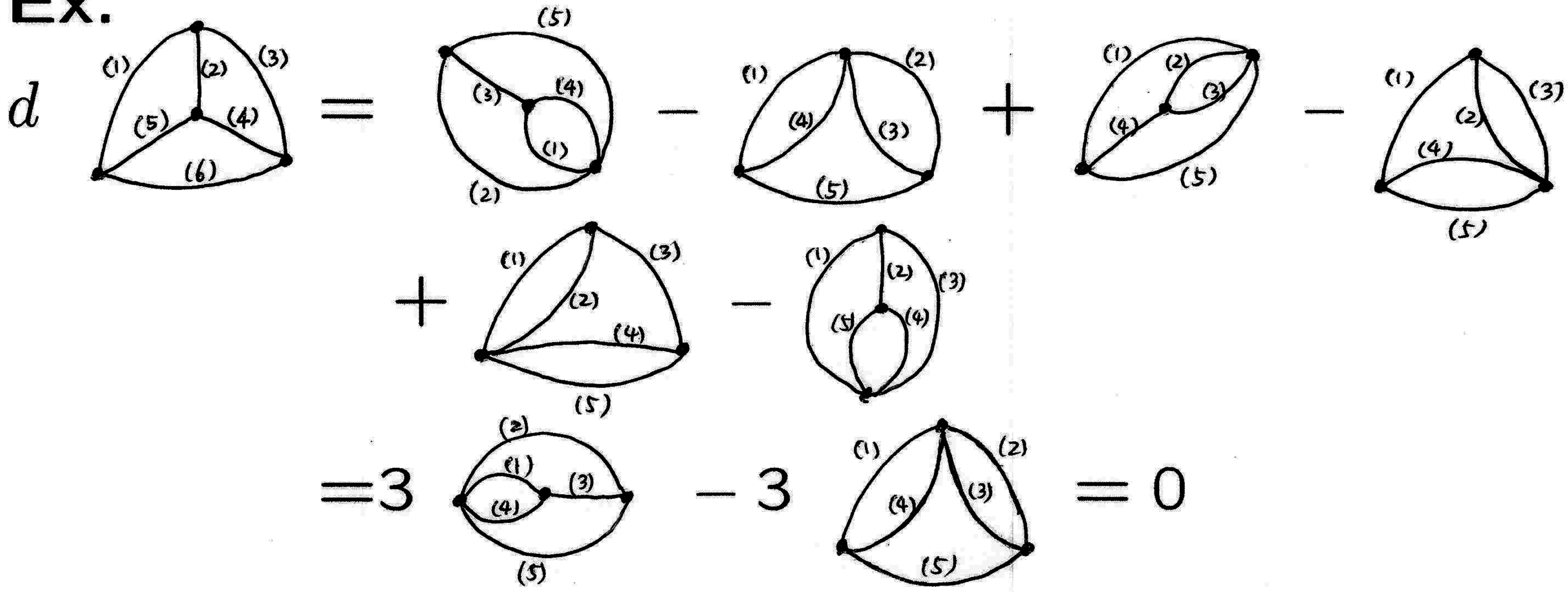
$\mathcal{G} = \mathrm{span}_{\mathbf{Q}}\{(\Gamma, o) \mid \Gamma : \text{graph}\}/(\Gamma, -o) \sim -(\Gamma, o)$

$d : \mathcal{G} \to \mathcal{G}$

$$d(\Gamma, o) = \sum_{e \in \mathrm{Edges}(\Gamma)} (\Gamma/e, \text{induced ori})$$

$(o = e_1 \wedge \cdots \wedge e_m,\ e = e_i$
$\Rightarrow (-1)^{i-1} e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_m)$

Ex.



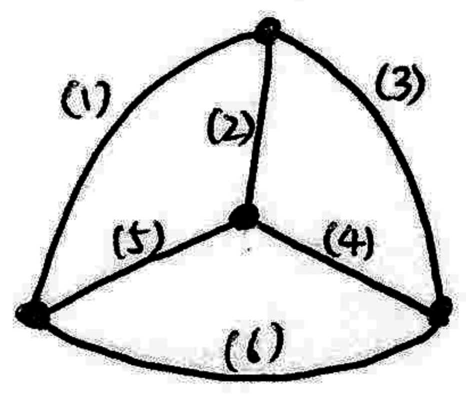
1. Graph homology

Lem. $d \circ d = 0$.

Def. (Graph homology)

$$H(\mathcal{G}) = \operatorname{Ker} d / \operatorname{Im} d$$
$$\left(= \bigoplus_{p,q} H_{p,q}(\mathcal{G})\right)$$

$$p = |\mathrm{Edges}(\Gamma)| - |\mathrm{Vert}(\Gamma)| = -\chi(\Gamma),$$
$$q = 2|\mathrm{Edges}(\Gamma)| - 3|\mathrm{Vert}(\Gamma)| \text{ (excess)}$$

Ex. $\gamma =$ 

is a nontrivial cycle.

$$H_{2,0}(\mathcal{G}) = \langle [\gamma] \rangle.$$

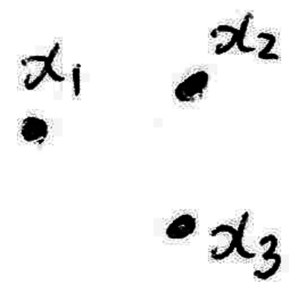
1. Graph homology

Table of dim $H_{p,q}(\mathcal{G})$
by D. Bar-Natan, B. McKay

q \ p	4	5	6	7	8	9
0	0	1	0	0	0	1
1	0	0	0	0	?	?
2	1	0	0	1	?	?
3		0	0	0	?	?
4		0	1	0	?	?
5			0	0	?	?
6				0	1	?
7				0	0	?
8					0	?
9					0	?
10					0	?

in their unpublished paper.

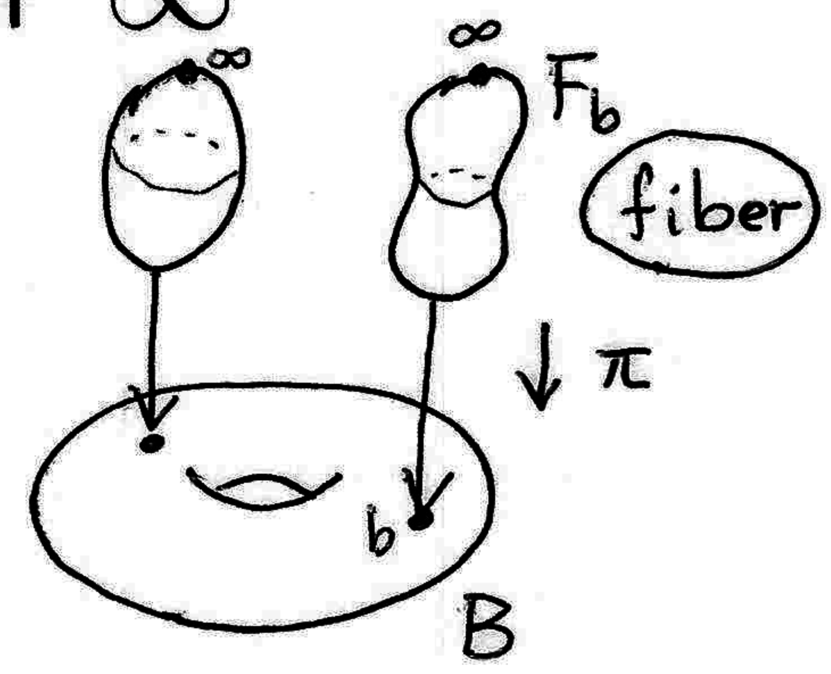
2. Kontsevich's map $I : H(\mathcal{G}) \to H^*(B)$



$\overline{C}_n(S^4)$ =Fulton-MacPherson compactification of $\{(x_1,\ldots,x_n) \in (S^4 \setminus \{\infty\})^n \mid i \neq j \Rightarrow x_i \neq x_j\}$

$\pi : E \to B$ $\quad S^4$-bundle trivialized near ∞

$$(E = \bigcup_{b \in B} F_b,\ F_b \cong S^4)$$

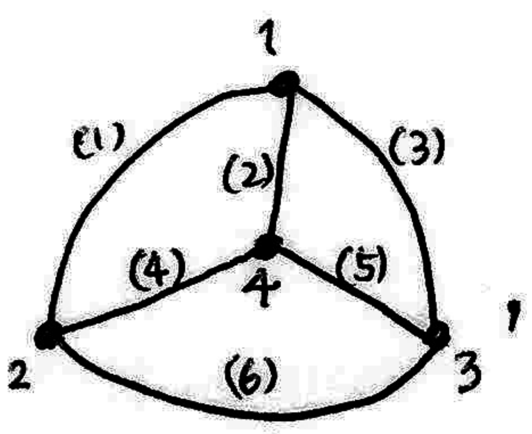


$\overline{C}_n(\pi) : \overline{C}_n E \to B$ $\quad \overline{C}_n(S^4)$-bundle

$$(\overline{C}_n E = \bigcup_{b \in B} \overline{C}_n(F_b))$$

$\exists\, \omega$: closed 3-form on $\overline{C}_2 E$ s.t.

$$H^3(\overline{C}_2(F_b); \mathbf{R}) = \langle [\omega|_{F_b}] \rangle$$

$\Gamma =$  ,

$$I(\Gamma) = \int_{\overline{C}_4(\text{fiber})} \underbrace{\omega_{12} \wedge \omega_{14} \wedge \omega_{13} \wedge \omega_{24} \wedge \omega_{34} \wedge \omega_{23}}_{18\text{-form}}$$

$\in \Omega^2(B)$ $\quad$ ↑ 16 dim

2. Kontsevich's map $I : H(\mathcal{G}) \to H^*(B)$

Theorem (Kontsevich 92)

$$I : \mathcal{G} \to \Omega^*(B); \quad \Gamma \mapsto I(\Gamma)$$

is a chain map, i.e., $dI(\Gamma) = (-1)^{|I(d\Gamma)|} I(d\Gamma)$. *Hence induces*

$$I : H(\mathcal{G}) \to H^*(B; \mathbf{R}).$$

Rem. B may be $B\mathrm{Diff}(D^4, \partial)$, the base of the universal bundle.

Main Theorem (W)

$$I : H(\mathcal{G}) \to H^*(B\mathrm{Diff}(D^4, \partial); \mathbf{R}).$$

is injective <u>*on trivalent part*</u>.

Outline of the proof (similar to odd-dim case)

Let $\gamma = \sum_{\Gamma} w_\Gamma \cdot \Gamma \in \mathcal{G}$ be a trivalent cycle.

Step 1. $\Gamma \mapsto (\exists\, \pi^\Gamma : E^\Gamma \to B)$ by surgery.

$\uparrow$ S^4-bundle

Step 2. Evaluation:

$$\langle I(\gamma), E^\Gamma \rangle = 2^{6k}(2k)!(3k)!\, w_\Gamma.$$

Thus,

$$\frac{1}{2^{6k}(2k)!(3k)!}\langle I(\gamma), \sum_{\Gamma} E^\Gamma \cdot \Gamma \rangle = \sum_{\Gamma} w_\Gamma \cdot \Gamma = \gamma.$$

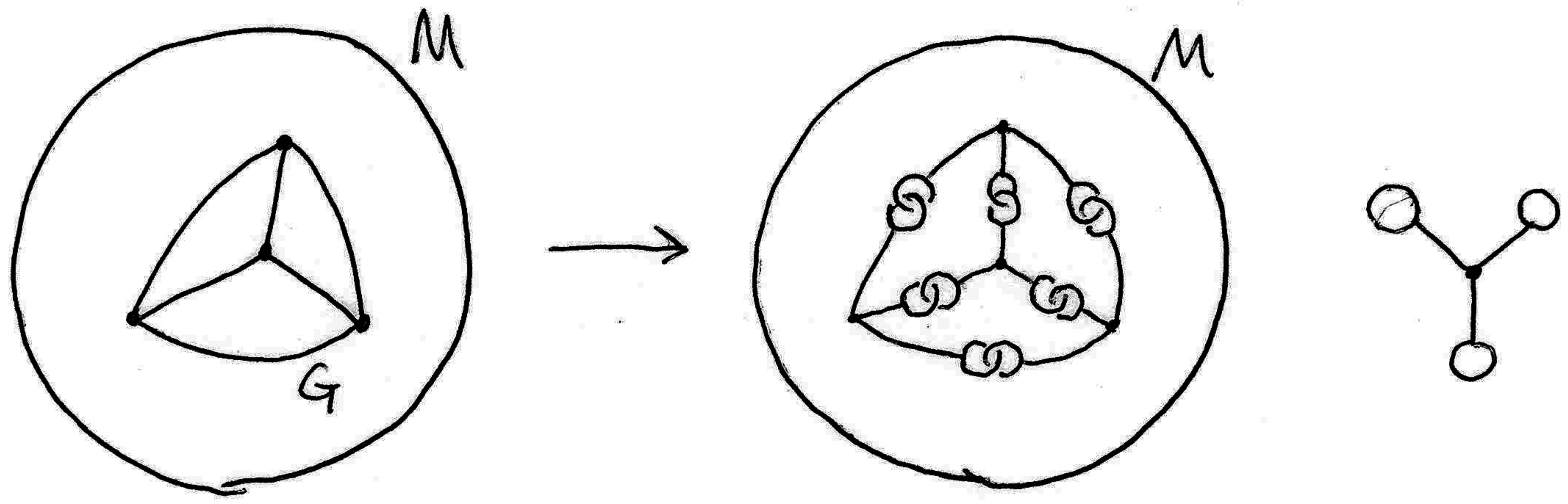
The composition

$$H_{k,0}(\mathcal{G}) \xrightarrow{I} H^*(B\mathrm{Diff}(D^4, \partial); \mathbf{R}) \underset{\times\, \mathrm{const}}{\xrightarrow{\sum_\Gamma E^\Gamma \cdot \Gamma}} H_{k,0}(\mathcal{G})$$

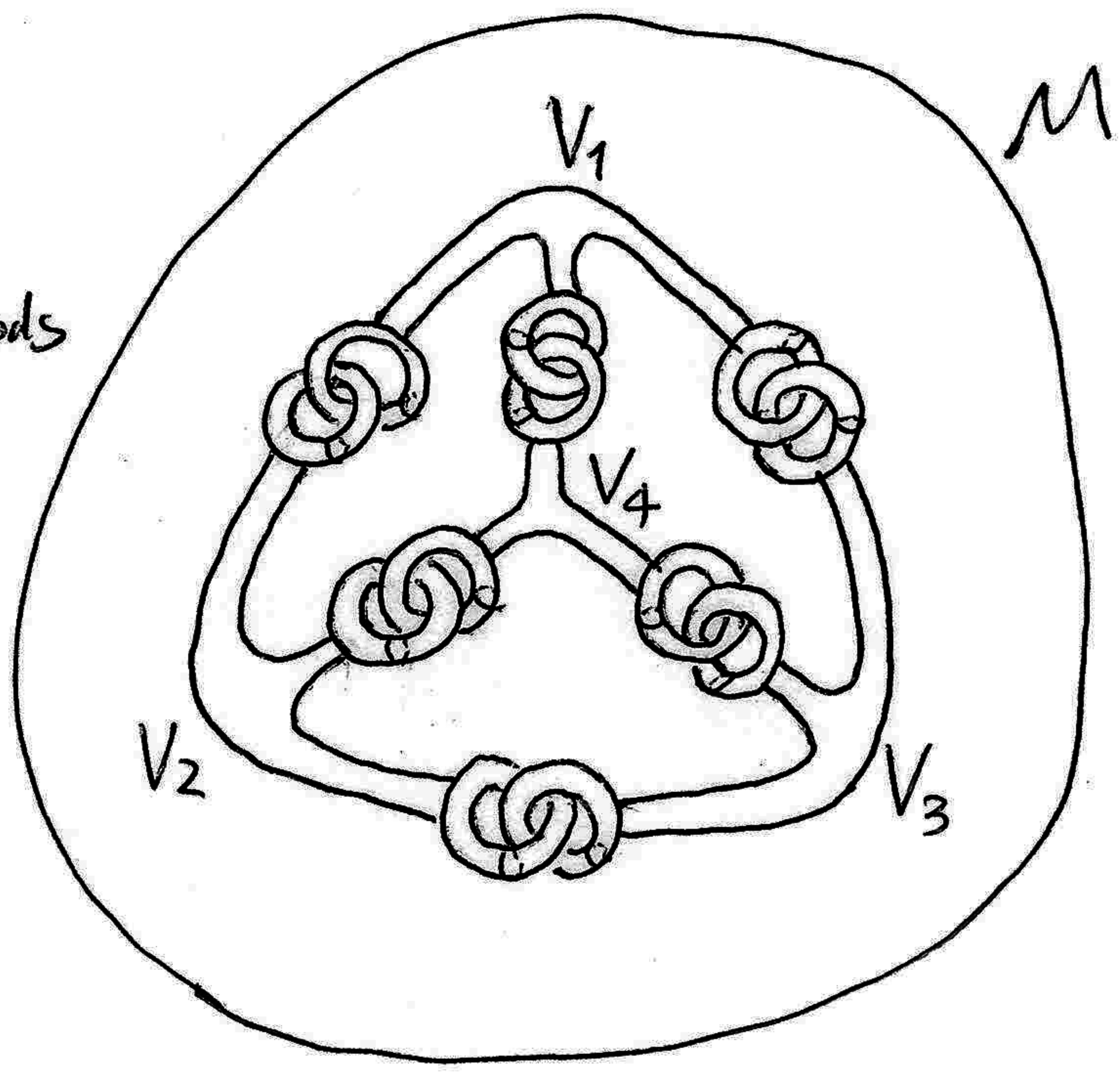
is the identity. □

3. Surgery construction

(3-dim, Goussarov-Habiro) Y-surgery



neighborhoods $\longrightarrow$

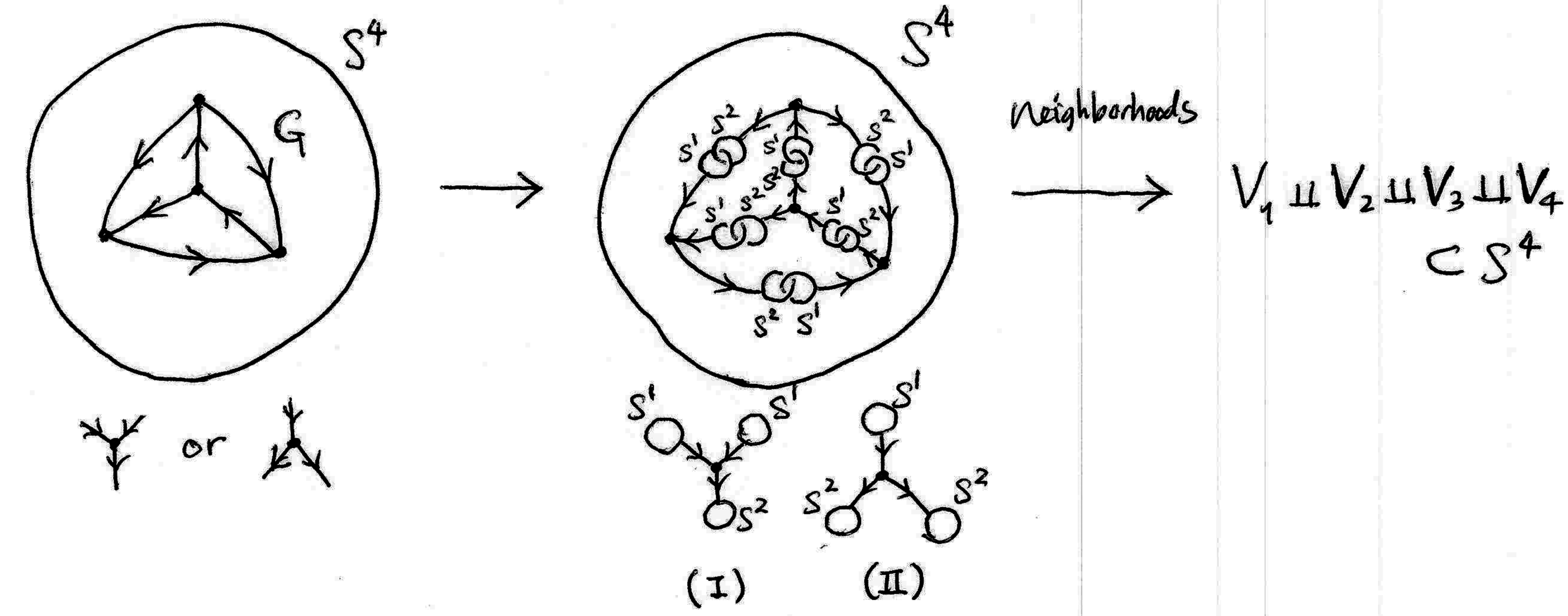


twist V_i along ∂V_i

$$\longrightarrow \quad M_G = \left(M - \bigcup_{i=1}^{2k} \operatorname{Int} V_i\right) \cup_{\varphi} \left(\bigcup_{i=1}^{2k} V_i\right)$$

3. Surgery construction

(4-dim analogue)



(family of) twists

$V_i = (\mathrm{I}) \Rightarrow s_i \in S^0 = \{-1, 1\}$, $\varphi_{s_i} : \partial V_i \overset{\cong}{\to} \partial V_i$

$V_i = (\mathrm{II}) \Rightarrow s_i \in S^1$, $\varphi_{s_i} : \partial V_i \overset{\cong}{\to} \partial V_i$

$\to$ family of surgeries $(S^4)_{\varphi_{s_1}, \dots, \varphi_{s_{2k}}} \cong S^4$
parametrized by $S^{p_1} \times \cdots \times S^{p_{2k}}$ $(p_i = 0, 1)$

$\to$ S^4-bundle $\pi^\Gamma : E^\Gamma \to S^{p_1} \times \cdots \times S^{p_{2k}}$

$$E^\Gamma = \bigcup_{(s_1, \dots, s_{2k})} (S^4)_{\varphi_{s_1}, \dots, \varphi_{s_{2k}}}$$

4. Evaluation – graph counting formula

Consider $\mathbf{R}^4$ instead of S^4.

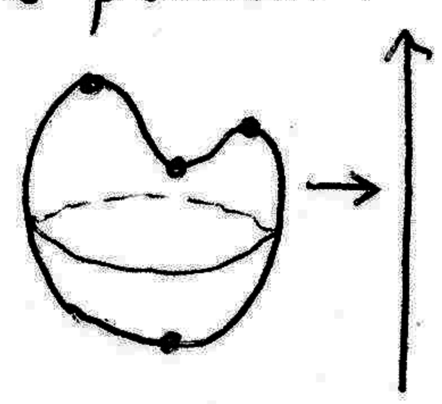
$\pi : E \to B$ $\quad (\mathbf{R}^4, \mathbf{R}^4 - \overset{\circ}{D^4})$-bundle

(i.e., standard outside D^4)

$f : E \to \mathbf{R}$ $\quad$ fiberwise Morse function

i.e., $f|_{F_b} : F_b \to \mathbf{R}$ Morse $\forall b \in B$

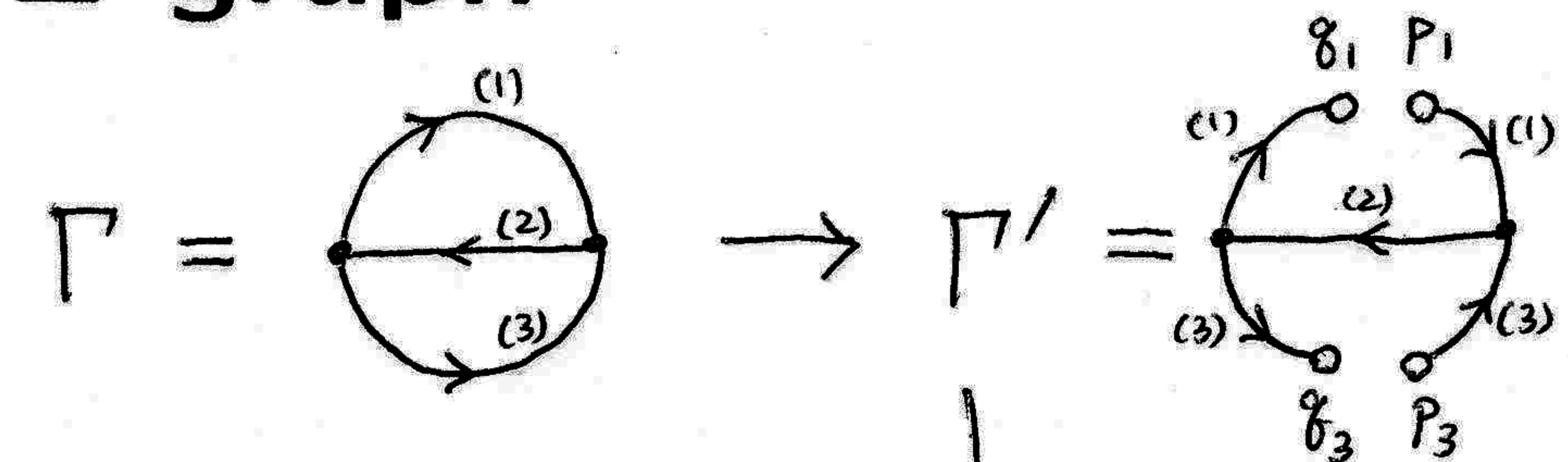
(Morse: generic height function with finitely many critical points.)



$\Rightarrow \xi$: (vertical) gradient of f along fiber

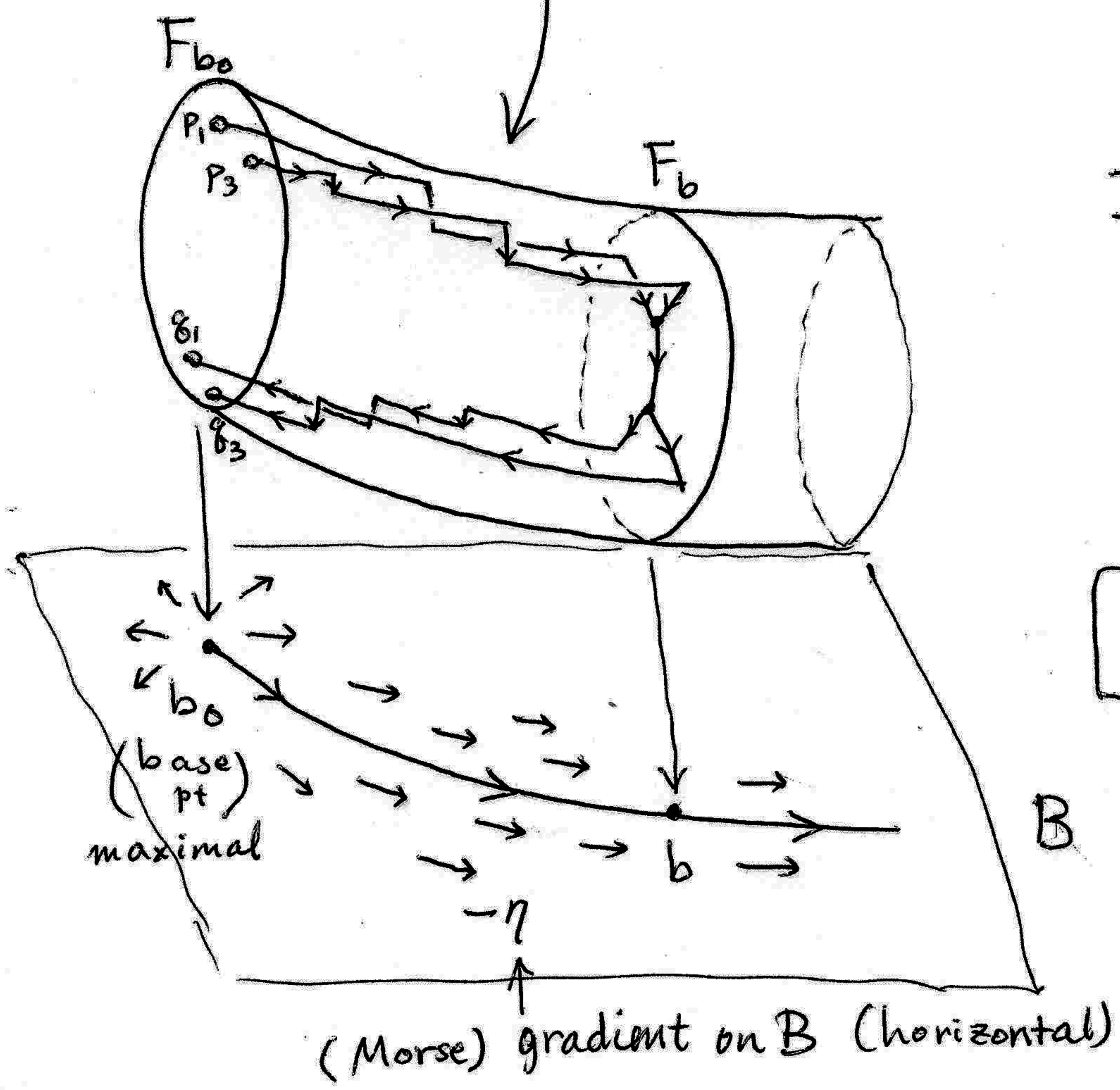
$\xrightarrow{\text{perturb}} \xi_1, \ldots, \xi_{3k}$

Z-graph



p_i, q_i: critical points of $f|_{F_{b_0}} : F_{b_0} \to \mathbf{R}$

ind p_i = ind q_i + 1



⇄ along critical locus

↓ along $-\xi_i$

Z - graph

4. Evaluation – graph counting formula

Z-graph can be counted if

- $\xi_1, \ldots, \xi_{3k}$ are generic.
- $\mathrm{ind}\, p_i = \mathrm{ind}\, q_i + 1$.

$\to n(\Gamma') \in \mathbf{Z}$

Lemma (Counting formula) *For $k \geq 2$,*

$$I'(\Gamma) = \mathrm{Tr}\Big(\sum_{\Gamma'} n(\Gamma')\, p_{i_1} \otimes g(q_{i_1}) \otimes \cdots \otimes p_{i_r} \otimes g(q_{i_r})\Big)$$
$$+ \textit{(correction term)}$$

(g: chain contraction for Morse complex $C_*(F_{b_0})$) satisfies

$$\langle I(\gamma), E^\Gamma \rangle = I'(\gamma)(E^\Gamma).$$

Rem. Analogue of T. Shimizu's identity Kontsevich "=" Fukaya.

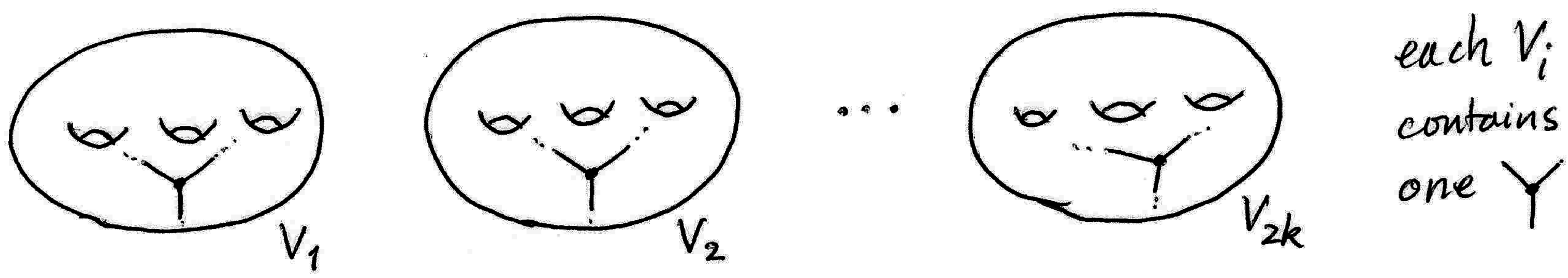
Rem. I utilizes framing on fiber, whereas I' doesn't. So we don't need to find a vertical framing on E^Γ compatible with V_i's.

Idea of the computation of $\langle I(\gamma), E^{\Gamma}\rangle$

By

- cancellation of intersection,
- dimensional reason,

I' only counts Z-graphs with



$\rightarrow$ explicitly countable.

$\rightarrow \langle I(\gamma), E^{\Gamma}\rangle = 2^{6k}(2k)!(3k)!\, w_{\Gamma}$. $\square$