The realization problem for Jørgensen numbers

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Background

- A Kleinian group is a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^n)$.
- $\mathrm{Isom}^+(\mathbb{H}^3) \cong PSL(2, \mathbb{C})$ (by the Poincaré extension).
- Σ : a complete hyperbolic manifold.
  $\Longrightarrow \exists \rho \ : \ \pi_1(\Sigma) \longrightarrow PSL(2, \mathbb{C})$: a discrete, faithful representation.
- Conversely, if $G < PSL(2, \mathbb{C})$ is a torsion-free Kleinian group, then $\mathbb{H}^3/G$ is a complete hyperbolic 3-manifold.
- The limit set $\Lambda(G)$ of a Kleinian group G is the set of accumulation points of $G \cdot z$ ($z \in \mathbb{H}^3$). $\Lambda(G) \subset \partial \mathbb{H}^3 = \hat{\mathbb{C}}$.
- The ordinary set is $\Omega(G) = \hat{\mathbb{C}} \setminus \Lambda(G)$. If $\Omega(G) \neq \emptyset$, then each component of $\Omega(G)/G$ is a Riemann surface.

Jørgensen's inequality

- When is a non-elementary group discrete (i.e. a Kleinian group)?

Theorem (Jørgensen '76)

$G = \langle f, g \rangle < PSL(2, \mathbb{C})$: a non-elementary Kleinian group. Then,

$$J(f, g) := |\mathrm{tr}^2(f) - 4| + |\mathrm{tr}(fgf^{-1}g^{-1}) - 2| \geq 1.$$

The constant 1 in the right-hand side is the best possible.

Remark (Jørgensen '76)

$G < PSL(2, \mathbb{C})$: a non-elementary group. Then,
G is a Kleinian group. $\Longleftrightarrow \forall f, g \in G$, $\langle f, g \rangle$ is a Kleinian group.

Jørgensen numbers

Definition

$G < PSL(2, \mathbb{C})$: a two-generator group.

- $J(G) := \inf\{J(f, g) \mid G = \langle f, g \rangle\}$ is the *Jørgensen number* of G.
- G is a *Jørgensen group* if G is Kleinian and $J(G) = 1$.

Example

- The modular group, the Picard group, the figure-eight knot group are Jørgensen groups. (Sato '01)
- For the Whitehead link group G_W, $J(G_W) = 2$. (Sato '04)
- For a quasi-fuchsian punctured torus group G, if there exists a generator (f, g) such that $\mathrm{tr}(f), \mathrm{tr}(g), \mathrm{tr}(fg) \in \mathbb{Z}$, then $J(G) = 9$. (Y. '16 master thesis)

Realization problem

Problem (Oichi-Sato '06)

For a real number $r \geq 1$, when does there exist a non-elementary Kleinian group whose Jørgensen number is equal to r?

Theorem (Yamashita-Y. '17)

For any $r \geq 1$, there exists a non-elementary Kleinian group G such that $J(G) = r$.

Known results

Theorem (Oichi-Sato '06)

- For any $n \in \mathbb{Z}_{\geq 1}$, there exists a non-elementary Kleinian group G such that $J(G) = n$.
- For any $r > 4$, there exists a classical Schottky group G such that $J(G) = r$.

Remark (Gilman '91, Sato' 98)

Any Schottky group is not a Jørgensen group.
In particular, if a Schottky group G is Fuchsian, then $J(G) > 4$.

Theorem (Callahan '09, PhD thesis)

The only torsion free Jørgensen group is the figure-eight knot group.

The Riley slice of the Schottky space

- Let $G_\rho = \langle X, Y_\rho \rangle$ be a non-elementary group generated by two parabolic transformations X, Y_ρ.
- We normalize so that $\mathrm{Fix}\,(X) = \{\infty\},\ \mathrm{Fix}\,(Y_\rho) = \{0\}$:

$$X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ Y_\rho = \begin{pmatrix} 1 & 0 \\ \rho & 1 \end{pmatrix}\ (\rho \in \mathbb{C})\,.$$

Definition

The Riley slice is defined by

$$\mathcal{R} := \left\{ \rho \in \mathbb{C} \,\middle|\, \begin{array}{l} G_\rho \text{ is free and Kleinian and} \\ \Omega\,(G_\rho)\,/\,G_\rho \text{ is a 4-times punctured sphere} \end{array} \right\}.$$

Remark (cf. Maskit-Swarup '88, Maskit '81)

The Riley slice $\mathcal{R}$ is on the boundary of the Schottky space.

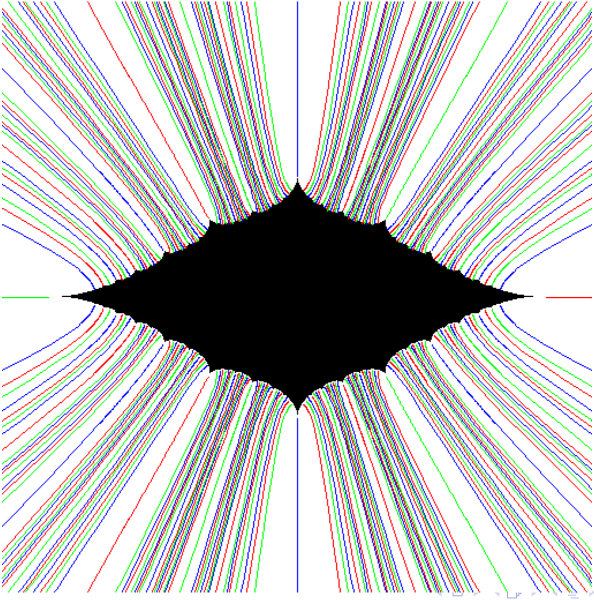
Keen-Series's theory for the Riley slice :

- a free homotopy class of a simple loop of slope $p/q \in \mathbb{Q}$ on the (topological) 4-times punctured sphere
  $\longleftrightarrow$ a representative word $V_{p/q}(\rho)$ in the conjugacy class in G_ρ
- tr $V_{p/q}(\rho)$ is a polynominal of ρ. (trace polynominal)
- Riley slice coincides with the closure of the union of rational pleating rays $\mathcal{P}_{p/q}$, that are disjoint curves running from distinct points on $\partial\mathcal{R}$ to ∞.

Remark (Ohshika-Miyachi '10)

G_ρ is free and Kleinian $\Longleftrightarrow \rho \in \overline{\mathcal{R}}$.

- Produced by software "OHT" from Professor Yasushi Yamashita



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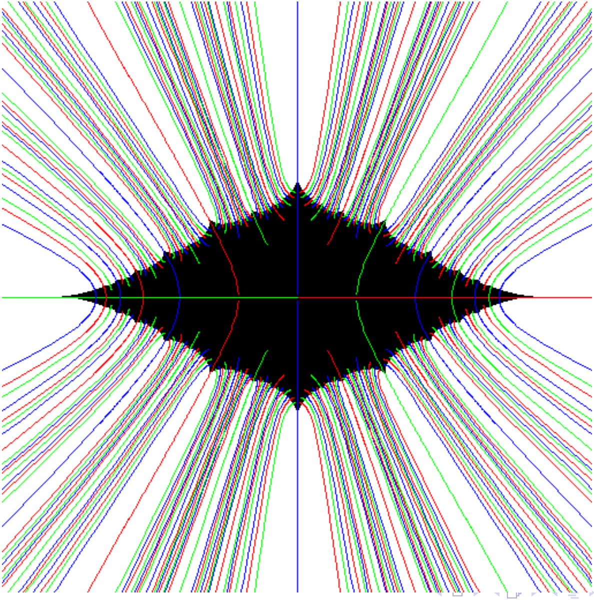


An extension of Oichi-Sato's theorem

Lemma

For any $\rho \in \overline{\mathcal{R}}$, $J(G_\rho) = |\rho|^2$.

Proposition (Keen-Series '94, Komori-Series '98)

For the end point $\rho \in \partial\mathcal{R}$ of pleating ray $\mathcal{P}_{p/q}$, $\mathrm{tr}\, V_{p/q}(\rho) = -2$.

Hence, we obtain the following theorem :

Theorem (Y. '16 master thesis)

For any $r \geq 2.467$, there exists a non-elementary Kleinian group G on the boundary of the Schottky space of rank 2 such that $J(G) = r$.

Contents

Line matrices

Definition

$[z, w]$: the geodesic in $\mathbb{H}^3$ from z to w for $z, w \in \hat{\mathbb{C}}$.

$M([z,w]) := \dfrac{i}{w-z}\begin{pmatrix} z+w & -2zw \\ 2 & -z-w \end{pmatrix}$: π-rotation about $[z, w]$.

For $a \ge 1$,

$$P_a := M([a, -3a]), Q := M([1, -1]), R := M([0, \infty]).$$

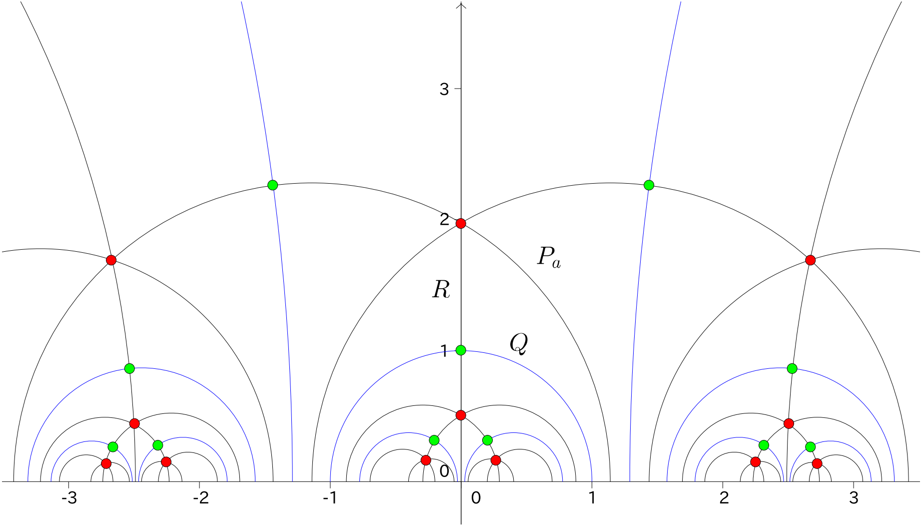
Lemma

For $a \ge 1$, $G_a := \langle P_a, Q, R \rangle$ is a non-elementary Kleinian group.

Proof.

Since G_a has a presentation
$\langle P_a, Q, R \mid (P_a)^2 = Q^2 = R^2 = (QR)^2 = (RP_a)^3 = 1 \rangle$,
it acts $\mathbb{H}^2$ properly discontinously as the Poincaré extension. □

- A fundamental polygon for $G_a \curvearrowright \mathbb{H}^2$



- The diagonal slice of singular solid torus (studied by Series-Tan-Yamashita)

Let $A_a := P_a Q = \begin{pmatrix} -3a/2 & 1/2 \\ -1/2 & -1/2a \end{pmatrix}$, $B := R = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

- Since $Q = A_a B A_a^{-1} B^{-1} A_a B$, We have $G_a = \langle A_a, B \rangle < PSL(2, \mathbb{C})$.
- For any $a \geq 1$, $\mathrm{tr}[A_a, B] = 1$.

Proposition

If $1 \leq a \leq \dfrac{\sqrt{7}+2}{3}$, then $J(G_a) = J(A_a, B) = \dfrac{(3a^2-1)^2}{4a^2}$.

In particular...

Theorem (Yamashita-Y. '17)

For any $1 \leq r \leq 4$, there is a non-elementary Kleinian group G such that $J(G) = r$. Hence, we obtained a complete solution for realization problem.

Jørgensen groups of parabolic type

Definition

A non-elementary Kleinian group G is *extreme* if there exists a generating pair (f, g) such that $J(f, g) = 1$.

if extreme, then Jørgensen group.

Theorem (Jørgensen-Kiikka '75)

If G is a extreme Fuchsian group, then G is a triangle group of signature $(2, 3, q)$ $(7 \leq q \leq \infty)$.

Proposition

Let $G = \langle f, g \rangle$ be a extreme Kleinian group of parabolic type (i.e. f is parabolic). Then, up to conjugation,

$$f = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; g = g_{\sigma,\mu} := \begin{pmatrix} \mu\sigma & \mu^2\sigma - 1/\sigma \\ \sigma & \mu\sigma \end{pmatrix} \quad (|\sigma| = 1, \, \mu \in \mathbb{C}).$$

Theorem (Li-Oichi-Sato '04, '05)

In the above normalization, extreme Kleinian groups of parabolic type in the case $\mu = ik$ $(k \in \mathbb{R})$ are completely classified.

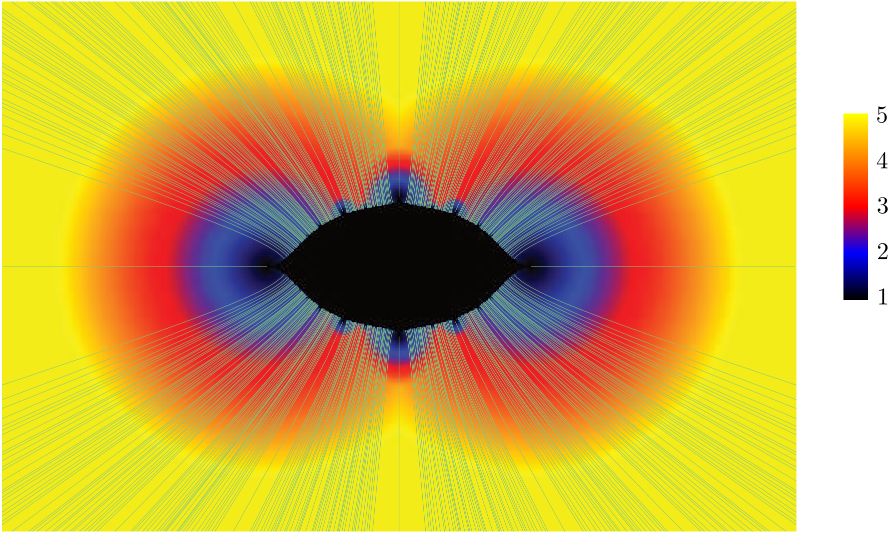
Conjecture (Oichi-Sato)

For any Jørgensen group G, there exists (σ, ik) $(|\sigma| = 1, k \in \mathbb{R})$ such that G is conjugate to $G_{\sigma,ik} = \langle f, g_{\sigma,ik} \rangle$.

Theorem (Callahan '09, PhD thesis)

$PGL(2, \mathcal{O}_3)$, $PSL(2, \mathcal{O}_3)$, $PSL(2, \mathcal{O}_7)$, $PSL(2, \mathcal{O}_{11})$ $(\mathcal{O}_d := \mathbb{Q}(\sqrt{-d}))$ are counterexamples for the conjecture.

- Jørgensen numbers on the diagonal slice



Thank you for your attention!