# Another proof to Kotschick-Morita's Theorem (tentative title)

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This article was a part of the draft "*An affirmative answer to a conjecture for Metoki class*" by Kentaro Mikami, submitted to some Journal. In order to shorten the draft, the author will put here one of the two proofs of the next theorem by D. Kotschick and S. Morita ([4]).
In this note, we just write down the proof by Groebner Basis theory.
About mathematical background, we refer to [4] or the draft "*An affirmative answer to a conjecture for Metoki class*" by Kentaro Mikami. For more precise notations or notions, we refer to [6].
We use Maple Groebner Package for computing Groebner Basis and the normal form.
There are several symbol calculus softwares beside Maple, Mathematica, Risa/Asir and so on. Risa/Asir is popular in Japanese mathematicians because it is bundled in Math Libre Disk which is distributed on the annual meetings of Mathematical Society of Japan. So, the author uploads the source code and the output about Risa/Asir concerning to the Theorem below by D. Kotschick and S. Morita on URL http://www.math.akita-u.ac.jp/˜mikami/Conj4MetokiClass/. You can compare the results by Maple (which is on this paper) and those on the same URL, then you will understand that the both are the same, up to non-zero scalar multiples.

There are two kinds of cohomology groups $\mathrm{H}^\bullet_{\mathrm{GF}}(\mathfrak{ham}_{2n}, \mathfrak{sp}(2n, \mathbb{R}))_w$ and $\mathrm{H}^\bullet_{\mathrm{GF}}(\mathfrak{ham}^0_{2n}, \mathfrak{sp}(2n, \mathbb{R}))_w$.
When $n = 1$, Gel'fand-Kalinin-Fuks ([2]) showed that $\mathrm{H}^\bullet_{\mathrm{GF}}(\mathfrak{ham}_2, \mathfrak{sp}(2, \mathbb{R}))_w = 0$ for the weight $w = 2, 4, 6$ and the $\mathrm{H}^7_{\mathrm{GF}}(\mathfrak{ham}_2, \mathfrak{sp}(2, \mathbb{R}))_8 \cong \mathbb{R}$ whose generator is called the Gel'fand-Kalinin-Fuks class.
The next non-trivial result in this context is $\mathrm{H}^9_{\mathrm{GF}}(\mathfrak{ham}_2, \mathfrak{sp}(2, \mathbb{R}))_{14} \cong \mathbb{R}$, which is proved by S. Metoki ([5]) in 1999. Here, $\mathfrak{ham}_2$ denotes the Lie algebra of the formal Hamiltonian vector fields on $\mathbb{R}^2$.
D. Kotschick and S. Morita ([4]) studied $\mathrm{H}^\bullet_{\mathrm{GF}}(\mathfrak{ham}^0_2, \mathfrak{sp}(2, \mathbb{R}))_w$ and determined the whole space for $w \leq 10$, where $\mathfrak{ham}^0_2$ is the Lie subalgebra of the formal Hamiltonian vector fields which vanish at the origin of $\mathbb{R}^2$. One of their several results is:

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**Theorem**[4]There is a unique element $\eta \in \mathrm{H}^5_{\mathrm{GF}}(\mathfrak{ham}^0_2)^{Sp}_{10} \cong \mathbb{R}$ such that

$$\text{Gel'fand-Kalinin-Fuks class } = \eta \wedge \omega \in \mathrm{H}^7_{\mathrm{GF}}(\mathfrak{ham}_2, \mathfrak{sp}(2, \mathbb{R}))_8$$

where $\omega$ is the cochain associated with the linear symplectic form of $\mathbb{R}^2$.

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We do not want to repeat what is the conjecture by D. Kotschick and S. Morita ([4]). The draft "*An affirmative answer to a conjecture for Metoki class*" studied this conjecture.
Our aim of this rough draft is to give another proof of the theorem above by using Gröbner basis theory (cf. [1]).

Let $x, y$ be the standard basis of $\mathbb{R}^2$ with the Poisson bracket is $\{x, y\} = 1$. We denote the standard basis of $A$-homogeneous polynomials of $x$ and $y$ as $\dfrac{x^a}{a!}\dfrac{y^{A-a}}{(A-a)!}$ and the dual basis is written by $z^a_A$.

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Using the method in [6], we can understand the structures of $\mathrm{C}^{\bullet}_{\mathrm{GF}}(\mathfrak{ham}^1_2, \mathfrak{sp}(2,\mathbb{R}))_{10}$ concretely. We denote $\mathrm{C}^{\bullet}_{\mathrm{GF}}(\mathfrak{ham}^1_2, \mathfrak{sp}(2,\mathbb{R}))_{10}$ by $C^{\bullet}$. We choose our concrete bases as $\{\mathbf{q}_i\}_{i=1}^{9}$ of $C^4$, $\{\mathbf{w}_i\}_{i=1}^{12}$ of $C^5$, and $\{\mathbf{r}_i\}_{i=1}^{4}$ of $C^6$. Then the matrix representations of linear maps $d_1 : C^4 \to C^5$ and $d_1 : C^5 \to C^6$ are given as

$$[d_1(\mathbf{q}_1), \ldots, d_1(\mathbf{q}_9)] = [\mathbf{w}_1, \ldots, \mathbf{w}_{12}]M$$

and

$$[d_1(\mathbf{w}_1), \ldots, d_1(\mathbf{w}_{12})] = [\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4]N$$

where

$$ {}^tM = \begin{bmatrix} -\frac{135}{4} & 0 & -60 & \frac{15}{2} & -45 & -15 & \frac{5}{4} & -\frac{45}{4} & \frac{75}{2} & 0 & 0 & 0 \\ \frac{108}{11} & \frac{18}{11} & 0 & 0 & 0 & \frac{60}{11} & \frac{46}{11} & -\frac{90}{11} & \frac{156}{11} & 0 & 0 & 0 \\ \frac{27}{4} & 0 & 12 & -\frac{9}{2} & -9 & 0 & 0 & 0 & 0 & \frac{27}{4} & 18 & 0 \\ 0 & 0 & -10 & \frac{2}{3} & -2 & 2 & 1 & 0 & 6 & 4 & -1 & 0 \\ 0 & \frac{5}{2} & 29 & \frac{47}{3} & -23 & 43 & \frac{13}{2} & \frac{9}{2} & 25 & 16 & -\frac{71}{2} & 0 \\ 0 & 5 & 45 & \frac{155}{6} & -40 & 65 & 10 & 0 & 50 & 20 & -\frac{115}{2} & 0 \\ 0 & \frac{3}{2} & 18 & \frac{23}{2} & -3 & 30 & \frac{11}{2} & \frac{9}{2} & 9 & 6 & -33 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 7 & 0 & -6 & 0 & 0 & 0 \\ 0 & 0 & -6 & -3 & 0 & 0 & 0 & 0 & 0 & 3 & -6 & 70 \end{bmatrix} \tag{1}$$

and

$$N = \begin{bmatrix} 0 & 140 & 0 & 0 & 0 & -15 & 15 & 30 & \frac{5}{2} & 0 & 0 & 0 \\ -5 & -4 & \frac{1}{4} & -\frac{11}{2} & \frac{31}{12} & \frac{31}{6} & -3 & -2 & \frac{5}{3} & -1 & 2 & 0 \\ -16 & 32 & -2 & -12 & \frac{22}{3} & \frac{58}{3} & -18 & -12 & -\frac{5}{3} & 0 & 8 & 0 \\ 0 & 0 & 0 & 42 & 7 & 0 & 0 & 0 & 0 & 0 & 14 & 3 \end{bmatrix} \tag{2}$$

Since $\mathrm{rank}M = 7$ and $\mathrm{rank}N = 4$, we see the dimensions of $d_1(C^4)$ and $\ker(d_1 : C^5 \to C^6)$ , and so on. The precise data of the structures of $\mathrm{C}^{\bullet}_{\mathrm{GF}}(\mathfrak{ham}^1_2, \mathfrak{sp}(2,\mathbb{R}))_{10}$ and $\mathrm{H}^{\bullet}_{\mathrm{GF}}(\mathfrak{ham}^1_2, \mathfrak{sp}(2,\mathbb{R}))_{10}$ is in the table below, where dim and rank mean the dimension of $C^{\bullet}$ and the rank of $d_1 : C^{\bullet} \to C^{\bullet+1}$, and Betti num is the Betti number, which is the dimension of the cohomology group $\mathrm{H}^{\bullet}_{\mathrm{GF}}(\mathfrak{ham}^1_2, \mathfrak{sp}(2,\mathbb{R}))_{10}$.

| $\mathfrak{ham}^1_2$, w=10 | $\mathbf{0}$ | $\to$ | $C^2$ | $\to$ | $C^3$ | $\to$ | $C^4$ | $\to$ | $C^5$ | $\to$ | $C^6$ | $\to$ | $\mathbf{0}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| dim | | | 1 | | 3 | | 9 | | 12 | | 4 | | |
| rank | | 0 | | 1 | | 2 | | 7 | | 4 | | 0 | |
| Betti num | | | 0 | | 0 | | 0 | | 1 | | 0 | | |

We also know the structures of $\mathrm{C}^{\bullet}_{\mathrm{GF}}(\mathfrak{ham}^0_2, \mathfrak{sp}(2,\mathbb{R}))_8$ well. For simplicity, we denote $\mathrm{C}^{\bullet}_{\mathrm{GF}}(\mathfrak{ham}^0_2, \mathfrak{sp}(2,\mathbb{R}))_8$ by $\mathfrak{C}^{\bullet}$.

For instance, the matrix representations of $d_0 : \mathfrak{C}^6 \to \mathfrak{C}^7$ and $d_0 : \mathfrak{C}^7 \to \mathfrak{C}^8$ are given as follows:

$$
{}^t\overline{M} = \begin{bmatrix}
\frac{135}{4} & 0 & -20 & -60 & 15 & \frac{5}{4} & -\frac{135}{4} & -15 & -15 & 0 & 0 & 0 & 0 & 0 \\
-18 & -12 & 0 & 0 & 0 & \frac{23}{3} & -45 & 21 & 10 & 0 & 0 & 0 & 0 & 0 \\
-\frac{27}{4} & 0 & -20 & 12 & 3 & 0 & 0 & 0 & 0 & \frac{63}{8} & -18 & 0 & 0 & 0 \\
0 & 0 & 2 & -10 & \frac{2}{3} & 1 & 2 & -\frac{1}{3} & 2 & -7 & 1 & 0 & 0 & 0 \\
0 & -\frac{11}{2} & -17 & 9 & \frac{71}{12} & \frac{7}{8} & \frac{51}{8} & \frac{233}{12} & 10 & -\frac{49}{4} & 4 & 0 & 0 & 0 \\
0 & -\frac{1}{3} & -9 & 5 & \frac{49}{18} & \frac{5}{12} & \frac{55}{4} & \frac{47}{9} & \frac{19}{3} & -\frac{28}{3} & -\frac{1}{6} & 0 & 0 & 0 \\
0 & -\frac{5}{2} & -20 & 0 & \frac{65}{12} & -\frac{15}{8} & -\frac{75}{8} & \frac{65}{12} & -5 & \frac{35}{4} & -\frac{25}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 7 & -18 & -9 & 6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & -6 & 0 & 0 & 0 & 0 & 0 & -\frac{21}{2} & 6 & 70 & 0 & 0 \\
-\frac{1}{2} & 2 & -2 & \frac{14}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -28 & -\frac{14}{3} \\
-\frac{5}{2} & 2 & 0 & \frac{16}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -28 & -\frac{28}{3} \\
0 & 8 & 0 & 0 & 0 & -3 & 0 & 1 & -6 & 0 & 0 & 0 & -112 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 15 & -6 & 0 & -3/2 & -3 & 0 & 48 & -14 \\
0 & 0 & 0 & 0 & 0 & -\frac{2}{3} & -1 & -1 & -2 & \frac{5}{2} & -1 & 0 & 4 & -\frac{14}{3} \\
0 & 0 & 0 & 0 & \frac{4}{3} & \frac{11}{3} & -45 & \frac{46}{3} & 0 & \frac{5}{2} & 11 & 0 & -136 & 14 \\
0 & 0 & 2 & 0 & -1 & -\frac{3}{2} & \frac{123}{8} & -4 & 0 & -\frac{21}{16} & -\frac{33}{16} & 0 & 42 & -\frac{21}{8} \\
0 & 0 & 0 & 2 & -\frac{1}{3} & -\frac{1}{2} & \frac{39}{8} & -\frac{4}{3} & 0 & \frac{7}{16} & -\frac{13}{16} & 0 & 14 & -\frac{35}{24} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -\frac{20}{3} & 0 & 2
\end{bmatrix} \quad (3)
$$

and

$$
\overline{N} = \begin{bmatrix}
0 & -35 & 0 & 0 & 0 & -30 & 0 & -15 & \frac{25}{2} & 0 & 0 & 0 & -\frac{5}{2} & 0 \\
11 & -9 & -\frac{39}{8} & -\frac{31}{8} & -\frac{61}{2} & -75 & -10 & -10 & \frac{85}{2} & -\frac{2}{3} & -\frac{20}{3} & 0 & -1 & -3 \\
-16 & 8 & 9 & 5 & 52 & 20 & \frac{8}{3} & -2 & -\frac{55}{3} & 0 & 8 & 0 & 1 & 4 \\
0 & 0 & \frac{63}{2} & \frac{21}{2} & 84 & 0 & 0 & 0 & 0 & 0 & -14 & 3 & 0 & 3
\end{bmatrix} \quad (4)
$$

Since $\mathrm{rank}\overline{M} = 9$ and $\mathrm{rank}\overline{N} = 4$, we see the dimensions of $d_0(\mathfrak{C}^6)$ and $\ker(d_0 : \mathfrak{C}^7 \to C^6)$ , and so on. The precise data of the structures of $\mathrm{C}^{\bullet}_{\mathrm{GF}}(\mathfrak{ham}^0_2, \mathfrak{sp}(2,\mathbb{R}))_8$ and $\mathrm{H}^{\bullet}_{\mathrm{GF}}(\mathfrak{ham}^0_2, \mathfrak{sp}(2,\mathbb{R}))_8$ are in the table below.

| $\mathfrak{ham}^0_2$, w=8 | $\mathbf{0}$ | $\to$ | $\mathfrak{C}^3$ | $\to$ | $\mathfrak{C}^4$ | $\to$ | $\mathfrak{C}^5$ | $\to$ | $\mathfrak{C}^6$ | $\to$ | $\mathfrak{C}^7$ | $\to$ | $\mathfrak{C}^8$ | $\to$ | $\mathbf{0}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| dim | | | 5 | | 13 | | 17 | | 18 | | 14 | | 4 | | |
| rank | | 0 | | 5 | | 8 | | 9 | | 9 | | 4 | | 0 | |
| Betti num | | | 0 | | 0 | | 0 | | 0 | | 1 | | 0 | | |

Since $\mathrm{H}^5_{\mathrm{GF}}(\mathfrak{ham}^1_2, \mathfrak{sp}(2,\mathbb{R}))_{10}$ and $\mathrm{H}^7_{\mathrm{GF}}(\mathfrak{ham}^0_2, \mathfrak{sp}(2,\mathbb{R}))_8$ are both 1-dimensional, if

$$\omega\wedge : \mathrm{H}^5_{\mathrm{GF}}(\mathfrak{ham}^1_2, \mathfrak{sp}(2,\mathbb{R}))_{10} \longrightarrow \mathrm{H}^7_{\mathrm{GF}}(\mathfrak{ham}^0_2, \mathfrak{sp}(2,\mathbb{R}))_8$$

is non-zero map, then it is an isomorphism.
We need to check if $\omega\wedge\ker(d_1) \subset d_0\,(\mathrm{C}^6_{\mathrm{GF}}(\mathfrak{ham}^0_2, \mathfrak{sp}(2,\mathbb{R}))_8)$ or not. For that purpose, choose a basis $\mathbf{k}_1, \ldots, \mathbf{k}_8$ of $\ker(d_1)$ and linear independent cochains $\mathbf{b}_1, \ldots, \mathbf{b}_9$ in $\mathrm{C}^6_{\mathrm{GF}}(\mathfrak{ham}^0_2, \mathfrak{sp}(2,\mathbb{R}))_8$ such that $d_0\,(\mathbf{b}_1), \ldots, d_0\,(\mathbf{b}_9)$ is a basis of $d_0\,(\mathrm{C}^6_{\mathrm{GF}}(\mathfrak{ham}^0_2, \mathfrak{sp}(2,\mathbb{R}))_8)$.
By taking matrix representation, we see that

$$\mathrm{rank}(\omega\wedge\mathbf{k}_1, \ldots, \omega\wedge\mathbf{k}_8, d_0\,(\mathbf{b}_1), \ldots, d_0\,(\mathbf{b}_9)) = 10 > 9$$

Thus, for an element, say $\mathbf{h}$, which represents the non-trivial cohomology class, we have to check if $\omega\wedge\mathbf{h}$ is absorbed in $d_0\,(\mathrm{C}^6_{\mathrm{GF}}(\mathfrak{ham}^0_2, \mathfrak{sp}(2,\mathbb{R}))_8)$ or not, namely, if $\omega\wedge\mathbf{h}$ realizes the non-trivial cohomology class in $\mathrm{H}^7_{\mathrm{GF}}(\mathfrak{ham}^0_2, \mathfrak{sp}(2,\mathbb{R}))_8$ or not. This is our strategy to complete the proof of Theorem.

**Another proof by Gröbner bases:** Since the both methodologies of using Gröbner bases in order to investigate the cohomology groups $\mathrm{H}^5_{\mathrm{GF}}(\mathfrak{ham}^1_2, \mathfrak{sp}(2,\mathbb{R}))_{10}$ or $\mathrm{H}^7_{\mathrm{GF}}(\mathfrak{ham}^0_2, \mathfrak{sp}(2,\mathbb{R}))_8$ are the same, we discuss in the case of $\mathrm{H}^5_{\mathrm{GF}}(\mathfrak{ham}^1_2, \mathfrak{sp}(2,\mathbb{R}))_{10}$ in detail and write down only the result for $\mathrm{H}^7_{\mathrm{GF}}(\mathfrak{ham}^0_2, \mathfrak{sp}(2,\mathbb{R}))_8$. In particular, we discuss the key issue where $\omega\wedge$ is involved, carefully.
Let $\{\mathbf{w}_1, \ldots, \mathbf{w}_{12}\}$ be the basis of $C^5$ and $\{\mathbf{q}_1, \ldots, \mathbf{b}_9\}$ be the basis of $C^4$ as before. From the matrix representation (1) of the coboundary operator $d_1$ of $C^4 \to C^5$, we define the linear functions

$$g_j(y) = \sum_{k=1}^{12} \lambda_{k\,j} y_k \qquad (j = 1, \ldots, 9)$$

where $(\lambda_{k\,j}) = M$ and $\{y_1, \ldots, y_{12}\}$ are the auxiliary variables.
Fixing a monomial order of polynomials induced, say $y_1 \succ \cdots \succ y_{12}$, we get the Gröbner basis $GB_e$ of the ideal generated by $\{g_j(y) \mid j = 1, \ldots, 9\}$. This corresponds to the non-zero rows of the elementary matrix of $M$ obtained by the elementary row operations for $M$. Thus, the cardinality of $GB_e$ is equal to the rank of $M$, namely, to $\dim(d_1\,(C^4))$ and $\{\widehat{g}(\mathbf{w}) \mid \widehat{g} \in GB_e\}$ gives a basis of $d_1\,(C^4)$ (cf. Proposition 3.1 in [1]). In our case,

$$\begin{aligned} GB_e = [\ & 21y_7 - 9y_8 - 18y_9 - 15y_{10} + 30y_{11} - 140y_{12}, \\ & 18y_6 + 9y_8 + 15y_{10} - 30y_{11} + 140y_{12}, \\ & 1512y_5 + 75y_8 - 900y_9 - 666y_{10} - 1461y_{11} + 3290y_{12}, \\ & 36y_4 - 3y_8 + 36y_9 - 18y_{10} + 57y_{11} - 770y_{12}, \\ & 72y_3 + 3y_8 - 36y_9 - 18y_{10} + 15y_{11} - 70y_{12}, \\ & 63y_2 - 327y_8 + 396y_9 - 258y_{10} - 660y_{11} + 3080y_{12}, \\ & 189y_1 - 12y_8 + 144y_9 + 99y_{10} + 390y_{11} - 1820y_{12}\ ] \end{aligned}$$

In general, the NormalForm of a given polynomial $g$ with respect to the Gröbner basis is the "smallest" remainder of $g$ modulo by the Gröbner basis.
For a linear function $L(y)$ of $y_1, \ldots, y_{12}$, that $L(\mathbf{w})$ belongs to $d_1\,(C^4)$ is equivalent to the NormalForm of $L(y)$ with respect to $GB_e$ is zero.

Let $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4\}$ be the basis of $C^6$ as before. The kernel space of $d_1 : C^5 \longrightarrow C^6$, whose element is given by $\sum_{j=1}^{12} c_j \mathbf{w}_j$ satisfying $\sum_{j=1}^{12} c_j d_1\,(\mathbf{w}_j) = \mathbf{0}$, is characterized by 4 linear functions, say $f_1(c), f_2(c), f_3(c), f_4(c)$ of $c_1, \ldots, c_{12}$ given by

$$[f_1(c), f_2(c), f_3(c), f_4(c)] = [c_1, \ldots, c_{12}]\,{}^tN$$

where $N$ is the matrix representing the operator $d_1 : C^5 \to C^6$ (This means we deal with the dual map ${d_1}^* : \left(C^5\right)^* \leftarrow \left(C^6\right)^*$). In our case,

$$
\begin{aligned}
f_1 &= 140c_2 - 15c_6 + 15c_7 + 30c_8 + \frac{5}{2}c_9\\
f_2 &= -5c_1 - 4c_2 + \frac{1}{4}c_3 - \frac{11}{2}c_4 + \frac{31}{12}c_5 + \frac{31}{6}c_6 - 3c_7 - 2c_8 + \frac{5}{3}c_9 - c_{10} + 2c_{11}\\
f_3 &= -16c_1 + 32c_2 - 2c_3 - 12c_4 + \frac{22}{3}c_5 + \frac{58}{3}c_6 - 18c_7 - 12c_8 - \frac{5}{3}c_9 + 8c_{11}\\
f_4 &= 42c_4 + 7c_5 + 14c_{11} + 3c_{12}
\end{aligned}
$$

By taking a monomial order, say $c_1 \succ \cdots \succ c_{12}$, we get the Gröbner basis $GB$ of the ideal $\langle f_1(c), f_2(c), f_3(c), f_4(c)\rangle$. In our case,

$$
\begin{aligned}
GB = [\, & 42c_4 + 7c_5 + 14c_{11} + 3c_{12},\\
& 42c_3 + 28c_5 - 114c_6 + 198c_7 + 228c_8 + 117c_9 - 48c_{10} + 4c_{11} + 6c_{12},\\
& 56c_2 - 6c_6 + 6c_7 + 12c_8 + c_9,\\
& 168c_1 - 112c_5 - 182c_6 + 126c_7 + 84c_8 - 35c_9 + 24c_{10} - 128c_{11} - 12c_{12}\,]
\end{aligned}
$$

The $GB$ gives a basis of the subspace $\left({d_1}^* : \left(C^5\right)^* \leftarrow \left(C^6\right)^*\right)((C^6)^*)$.

Consider the polynomial $h = \sum_{j=1}^{12} c_j y_j$ where $\{y_1, \ldots, y_{12}\}$ are the other auxiliary variables.

Proposition 3.3 in [1] says that the NormalForm of $h$ with respect to the Gröbner basis $GB$ is written as $\sum_{j\in J} c_j \tilde{f}_j(y)$ where $J$ is a subset of $\{1, 2, \ldots, 12\}$, $\tilde{f}_j(y)$ is linear in $\{y_1, \ldots, y_{12}\}$, the cardinality of $J$ is $\dim\ker(d_1)$, and $\{\tilde{f}_j(\mathbf{w}) \mid j \in J\}$ gives a basis of $\ker(d_1)$. We continue the discussion in our case, then we have

$$
\begin{aligned}
&\tilde{f}_1 = 0, \quad \tilde{f}_2 = 0, \quad \tilde{f}_3 = 0, \quad \tilde{f}_4 = 0,\\
&\tilde{f}_5 = \frac{2}{3}y_1 - \frac{2}{3}y_3 - \frac{1}{6}y_4 + y_5, && \tilde{f}_6 = \frac{13}{12}y_1 + \frac{3}{28}y_2 + \frac{19}{7}y_3 + y_6,\\
&\tilde{f}_7 = -\frac{3}{4}y_1 - \frac{3}{28}y_2 - \frac{33}{7}y_3 + y_7, && \tilde{f}_8 = -\frac{1}{2}y_1 - \frac{3}{14}y_2 - \frac{38}{7}y_3 + y_8,\\
&\tilde{f}_9 = \frac{5}{24}y_1 - \frac{1}{56}y_2 - \frac{39}{14}y_3 + y_9, && \tilde{f}_{10} = -\frac{1}{7}y_1 + \frac{8}{7}y_3 + y_{10},\\
&\tilde{f}_{11} = \frac{16}{21}y_1 - \frac{2}{21}y_3 - \frac{1}{3}y_4 + y_{11}, && \tilde{f}_{12} = \frac{1}{14}y_1 - \frac{1}{7}y_3 - \frac{1}{14}y_4 + y_{12}
\end{aligned}
$$

Again, fixing the monomial order of $\{y_j\}$, we get the Gröbner basis $GB_k$ of the ideal generated by $\tilde{f}_j$ ($j \in J$) as

$$
\begin{aligned}
GB_k = [\, & 3y_8 - 36y_9 - 72y_{10} - 3y_{11} + 14y_{12}, \quad 3y_7 - 18y_9 - 33y_{10} + 3y_{11} - 14y_{12},\\
& 18y_6 + 108y_9 + 231y_{10} - 21y_{11} + 98y_{12}, \quad 36y_5 + 27y_{10} - 33y_{11} + 70y_{12},\\
& 2y_4 - 5y_{10} + 3y_{11} - 42y_{12}, \quad 12y_3 + 9y_{10} + 3y_{11} - 14y_{12},\\
& 9y_2 - 504y_9 - 1158y_{10} - 141y_{11} + 658y_{12}, \quad 3y_1 - 3y_{10} + 6y_{11} - 28y_{12}\,]
\end{aligned}
$$

in our case.

The cohomology $\mathrm{H}^5_{\mathrm{GF}}(\mathfrak{ham}^1_2, \mathfrak{sp}(2,\mathbb{R}))_{10}$ corresponds to the Gröbner basis $GB_{k/e}$ of the ideal generated by the NormalForm of $\widehat{g} \in GB_k$ with respect to $GB_e$. In our case, this is given by

$$
GB_{k/e} = [\, 3y_8 - 36y_9 - 72y_{10} - 3y_{11} + 14y_{12}\,] \tag{5}
$$

**$\mathrm{H}^7_{\mathrm{GF}}(\mathfrak{ham}^0_2, \mathfrak{sp}(2,\mathbb{R}))_8$ case:** In the case of $\mathrm{H}^7_{\mathrm{GF}}(\mathfrak{ham}^0_2, \mathfrak{sp}(2,\mathbb{R}))_8$, we use the notations $\overline{GB}_k$, $\overline{GB}_e$ and $\overline{GB}_{k/e}$ for the Gröbner bases corresponding to the kernel, $d_0$-image and $\mathrm{H}^7_{\mathrm{GF}}(\mathfrak{ham}^0_2, \mathfrak{sp}(2,\mathbb{R}))_8$ respectively. The space $d_0(\mathfrak{C}^6)$ is characterized by the following Gröbner basis:

$$\begin{aligned}\overline{GB}_e = [\ & 3y_{10} - 3y_{11} - 20y_{12} + 6y_{14},\\ & 100y_8 + 36y_9 - 15y_{11} - 420y_{12} - 420y_{13} + 350y_{14},\\ & 300y_7 + 84y_9 - 135y_{11} - 980y_{12} + 420y_{13} + 350y_{14},\\ & 100y_6 + 204y_9 - 135y_{11} - 1380y_{12} - 180y_{13} + 750y_{14},\\ & 40y_5 - 12y_9 + 15y_{11} - 460y_{12} - 60y_{13} - 590y_{14},\\ & 4800y_4 + 84y_9 + 2565y_{11} + 6020y_{12} + 420y_{13} - 10850y_{14},\\ & 1600y_3 - 84y_9 + 1035y_{11} - 6020y_{12} - 420y_{13} - 5950y_{14},\\ & 400y_2 - 12y_9 - 195y_{11} - 1860y_{12} - 5660y_{13} + 950y_{14},\\ & 450y_1 + 24y_9 + 315y_{11} + 220y_{12} + 120y_{13} + 1250y_{14}\ ]\end{aligned}$$

The kernel space of $d_0 : \mathfrak{C}^7 \to \mathfrak{C}^8$ is generated by

$$\begin{aligned} f_1 =& -35c_2 - 30c_6 - 15c_8 + \frac{25}{2}c_9 - \frac{5}{2}c_{13}\\ f_2 =& 11c_1 - 9c_2 - \frac{39}{8}c_3 - \frac{31}{8}c_4 - \frac{61}{2}c_5 - 75c_6 - 10c_7\\ & - 10c_8 + \frac{85}{2}c_9 - \frac{2}{3}c_{10} - \frac{20}{3}c_{11} - c_{13} - 3c_{14}\\ f_3 =& -16c_1 + 8c_2 + 9c_3 + 5c_4 + 52c_5 + 20c_6 + \frac{8}{3}c_7\\ & - 2c_8 - \frac{55}{3}c_9 + 8c_{11} + c_{13} + 4c_{14}\\ f_4 =& \frac{63}{2}c_3 + \frac{21}{2}c_4 + 84c_5 - 14c_{11} + 3c_{12} + 3c_{14}\end{aligned}$$

and the kernel space of $d_0 : \mathfrak{C}^7 \to \mathfrak{C}^8$ is characterized by the following Gröbner basis.

$$\begin{aligned}\overline{GB}_k = [\ & 3y_{10} - 3y_{11} - 20y_{12} + 6y_{14},\quad 12y_9 + 495y_{11} + 3260y_{12} + 60y_{13} - 950y_{14},\\ & y_8 - 15y_{11} - 102y_{12} - 6y_{13} + 32y_{14},\quad 3y_7 - 36y_{11} - 238y_{12} + 70y_{14},\\ & 2y_6 - 171y_{11} - 1136y_{12} - 24y_{13} + 338y_{14},\quad 4y_5 + 51y_{11} + 280y_{12} - 154y_{14},\\ & 16y_4 - 3y_{11} - 56y_{12} - 14y_{14},\quad 16y_3 + 45y_{11} + 168y_{12} - 126y_{14},\\ & 4y_2 + 3y_{11} + 14y_{12} - 56y_{13},\quad 2y_1 - 3y_{11} - 28y_{12} + 14y_{14}\ ]\end{aligned}$$

thus, $\mathrm{H}^7_{\mathrm{GF}}(\mathfrak{ham}^0_2, \mathfrak{sp}(2,\mathbb{R}))_8$ is characterized by

$$\overline{GB}_{k/e} = [\ 12y_9 + 495y_{11} + 3260y_{12} + 60y_{13} - 950y_{14}\ ]$$

Take $h(y) = 3y_8 - 36y_9 - 72y_{10} - 3y_{11} + 14y_{12}$ from $GB_{k/e}$ of (5). Now $h(\mathbf{w})$ is in $\ker(d_1 : C^5 \to C^6) \setminus d_1(C^4)$. We express the following element

$$\omega \wedge h(\mathbf{w}) = z^0_1 \wedge z^1_1 \wedge h(\mathbf{w})$$

by the basis of $\mathfrak{C}^7$. We see that

$$\omega \wedge h(\mathbf{w}) = z^0_1 \wedge z^1_1 \wedge h(\mathbf{w}) = -9\overline{\mathbf{w}}_7 + 105\overline{\mathbf{w}}_{10} + 3\overline{\mathbf{w}}_{11} + 14\overline{\mathbf{w}}_{12} = \overline{h}(\overline{\mathbf{w}})$$

where $\overline{h} = -9y_7 + 105y_{10} + 3y_{11} + 14y_{12}$. The NormalForm of $\overline{h}$ with respect to $\overline{GB}_e$ is

$$\frac{63}{25}y_9 + \frac{2079}{20}y_{11} + \frac{3423}{5}y_{12} + \frac{63}{5}y_{13} - \frac{399}{2}y_{14}$$

and is not zero. This finishes the proof of the Theorem. ∎

**Remark** We emphasize that everything starts from the concrete bases of cochain complexes $\mathrm{C}^4_{\mathrm{GF}}(\mathfrak{ham}^1_2, \mathfrak{sp}(2, \mathbb{R}$ $\mathrm{C}^5_{\mathrm{GF}}(\mathfrak{ham}^1_2, \mathfrak{sp}(2, \mathbb{R}))_{10}$, $\mathrm{C}^6_{\mathrm{GF}}(\mathfrak{ham}^1_2, \mathfrak{sp}(2, \mathbb{R}))_{10}$, $\mathrm{C}^6_{\mathrm{GF}}(\mathfrak{ham}^0_2, \mathfrak{sp}(2, \mathbb{R}))_8$ and $\mathrm{C}^7_{\mathrm{GF}}(\mathfrak{ham}^0_2, \mathfrak{sp}(2, \mathbb{R}))_8$. Even though we make use of Gröbner Base theory or use of classical linear algebra argument, we are based on some concrete matrix representations.

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