Measuring Power of Generalised Definite Languages

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Outline

1. Background I: measurability (5 min.)
2. Background II: known properties (5 min.)
3. Main results (10 min.)
4. Conclusion (5 min.)

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Density of formal languages

The density of a language L over A is defined as

$$\delta_A(L) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \frac{\#(L \cap A^i)}{\#(A^i)}.$$

Here $\#(X)$ denotes the cardinality of X.

$\delta_A(L)$ can be regarded as the (average) **probability** that a randomly chosen word is in L.

Density of formal languages

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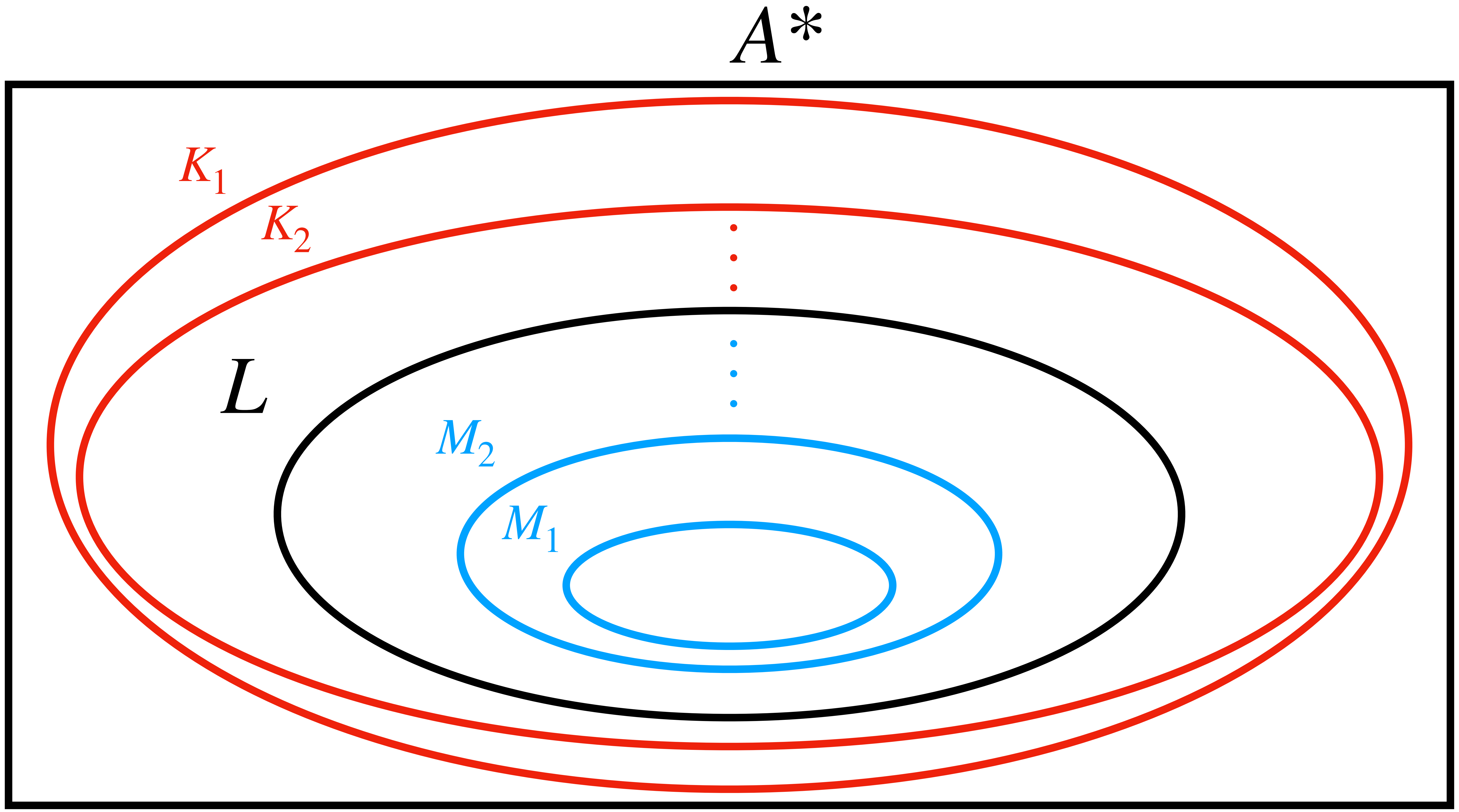
Example 1: $\delta_A((AA)^*) = \dfrac{1}{2}$.

Example 2: $\delta_A(A^*wA^*) = 1$ for any w.

Example 3: $L_\perp = \{w \in A^* \mid 3^n \leq |w| < 3^{n+1}$ for some even $n\}$
does **not** have a density.

Theorem (cf. [Berstel 1973]):
Every regular language *do have a rational density*.

$\mathscr{C}$-measurability [S., SOFSEM'21] (cf. [Buck, 1946])



L is said to be $\mathscr{C}$**-measurable** if there exists an *infinite sequence of pairs of languages* $(M_n, K_n)_{n\in\mathbb{N}}$ in $\mathscr{C}$ such that $M_n \subseteq L \subseteq K_n$ and $\lim_{n\to\infty} \delta_A(K_n \setminus M_n) = 0$.

Example of a regular measurable language

Theorem [S., SOFSEM'21]:

B $= \{w \in A \mid |w|_a = |w|_b\}$ over $A = \{a, b\}$ is regular measurable.

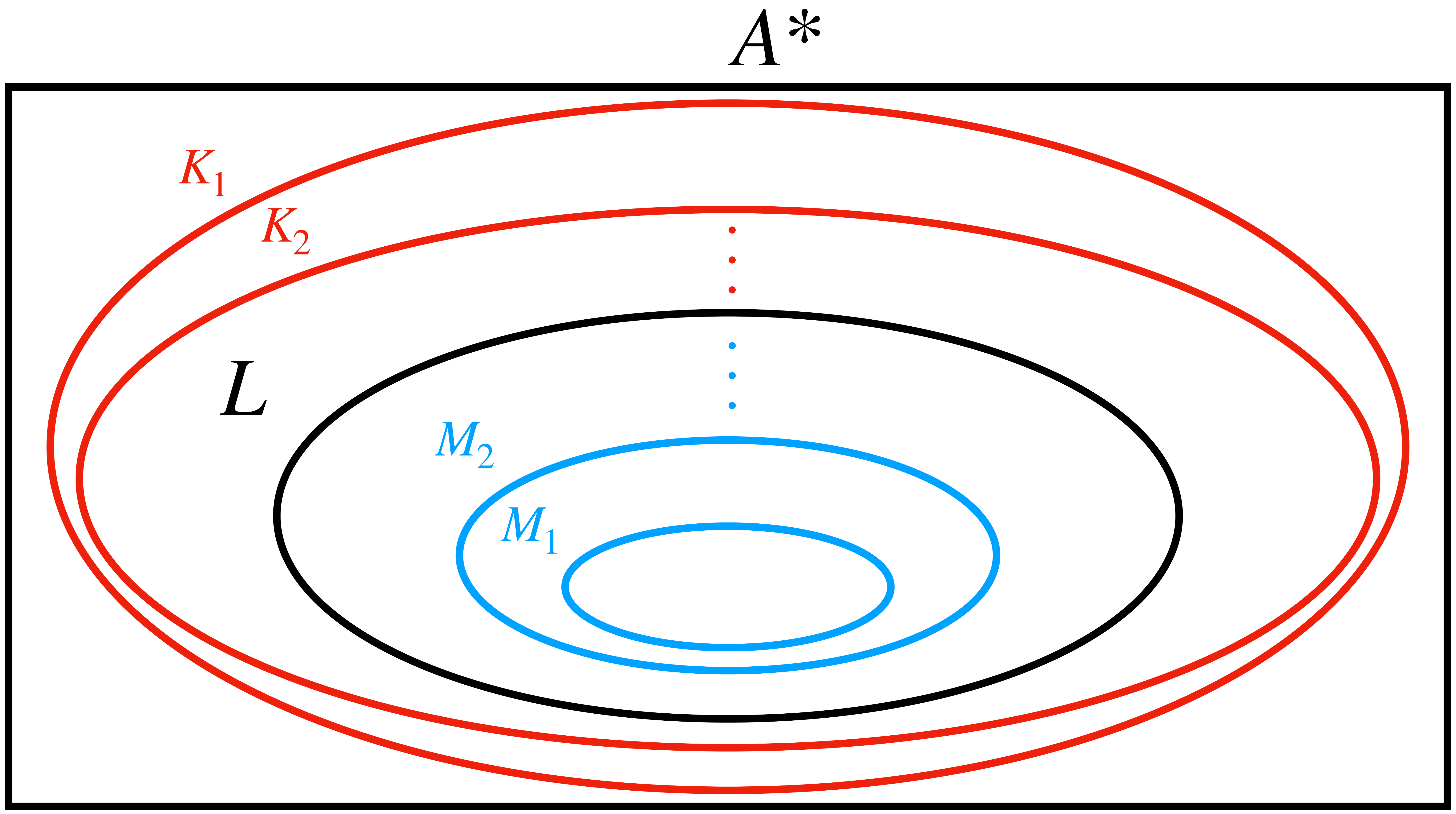
the # of occurrences of a in w

Proof: Let $L_k = \{w \in A^* \mid |w|_a = |w|_b \mod k\}$ for each $k \geq 1$.

Then, for each $k \geq 1$, B $\subseteq L_k$ and $\delta_A(L_k) = \dfrac{1}{k} \to 0$ (if $k \to \infty$).

Thus the infinite sequence $(\varnothing, L_k)_{k \geq 1}$ converges to B.

$\mathscr{C}$-measurability [S., SOFSEM'21] (cf. [Buck, 1946])



L is said to be $\mathscr{C}$**-measurable** if there exists an *infinite sequence of pairs of languages* $(M_n, K_n)_{n\in\mathbb{N}}$ in $\mathscr{C}$ such that $M_n \subseteq L \subseteq K_n$ and $\lim\limits_{n\to\infty} \delta_A(K_n \setminus M_n) = 0$.

Original motivation of mesurability

- A non-empty word w is said to be ***primitive*** if it can not be represented as a power of shorter words, i.e., $w = u^n \Rightarrow u = w$ (and $n = 1$).
  Q denotes the set of all primitive words over $\{a, b\}$.

Example : $ababa \in \mathrm{Q}$ $\quad ababab = (ab)^3 \notin \mathrm{Q}$

Primitive worsd conjecture [Dömösi-Horvath-Ito 1991]:
Q is *not* context-free.

My idea: while every context-free language is
regular measurable, Q is regular **im**measurable.

Summary of [S., SOFSEM'21]

Regular measurable languages

A deterministic CFL

M

$= \{w \in \{a,b\}^* \mid |w|_a > |w|_b\}$

Q

B

Many complex context-free languages.

There are **uncountably many** regular measurable languages.

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Some properties of $\mathscr{C}$-measurability [S. DLT'21]

Notation: $\mathscr{M}_A(\mathscr{C}) = \{L \subseteq A^* \mid L \text{ is } \mathscr{C}\text{-measurable}\}$

- $\mathscr{M}_A(\mathscr{C})$ can be defined as the *Carathéodory extension* of $\mathscr{C}$, a standard notion from measure theory.

- $\mathscr{M}_A(\mathscr{C})$ is closed under Boolean operations and left-and-right quotients if $\mathscr{C}$ is closed under these operations.

- "Is a given CFG generates a regular measurable languages?" is **undecidable**.

Some properties of $\mathscr{C}$-measurability [S. DLT'21]

Notation: $\mathscr{M}_A(\mathscr{C}) = \{L \subseteq A^* \mid L \text{ is } \mathscr{C}\text{-measurable}\}$

Q: How about the decidability of $\mathscr{C}$-measurability for some **subclass $\mathscr{C}$ of regular languages?**

- "Is a given CFG generates a regular measurable languages?" is **undecidable**.

Decidability

- PT-measurability for DFAs is **decidable in linear time** [SYN 2022], where PT is the class of all *piecewise testable* languages.

  Definition: L is ***piecewise testable*** [Simon 1972] if it can be represented as a finite Boolean combination of languages of the form $L_w = A^* a_1 A^* a_2 \ldots A^* a_n A^*$ where $w = a_1 a_2 \cdots a_n$.

Decidability

- PT-measurability for DFAs is **decidable in linear time** [SYN 2022], where PT is the class of all *piecewise testable* languages.
- AT-measurability for DFAs is **coNP-complete** [SYN 2022], where AT is the class of all *alphabet testable* languages.

Definition: L is ***alphabet testable*** if it can be represented as a finite Boolean combination of languages of the form A^*aA^* (where $a \in A$) .

Decidability

- PT-measurability for DFAs is **decidable in linear time** [SYN 2022], where PT is the class of all *piecewise testable* languages.
- AT-measurability for DFAs is **coNP-complete** [SYN 2022], where AT is the class of all *alphabet testable* languages.
- The decidability of LT-measurability for DFAs is **open** [S. DLT'22], where LT is the class of all *locally testable* languages.

  Definition: L is ***locally testable*** if it can be represented as a finite Boolean combination of languages of the form uA^*, A^*v, A^*wA^*.

Decidability

Notation: $\mathscr{M}_A(\mathscr{C}) = \{L \subseteq A^* \mid L \text{ is } \mathscr{C}\text{-measurable}\}$

- PT-measurability for DFAs is **decidable in linear time** [SYN 2022], where PT is the class of all *piecewise testable* languages.
- AT-measurability for DFAs is **coNP-complete** [SYN 2022], where AT is the class of all *alphabet testable* languages.
- The decidability of LT-measurability for DFAs is **open** [S. DLT'22], where LT is the class of all *locally testable* languages.
- Hierarchy is strict [S. DLT'22]: $\mathscr{M}_A(\mathsf{AT}) \subsetneq \mathscr{M}_A(\mathsf{PT}) \subsetneq \mathscr{M}_A(\mathsf{LT})$.

Decidability

Notation: $\mathscr{M}_A(\mathscr{C}) = \{L \subseteq A^* \mid L \text{ is } \mathscr{C}\text{-measurable}\}$

Main result of this work:

A **decidable** characterisation of LT-measurable regular languages.

- The decidability of LT-measurability for DFAs is **open** [S. DLT'22], where LT is the class of all *locally testable* languages.
- Hierarchy is strict [S. DLT'22]: $\mathscr{M}_A(\mathrm{AT}) \subsetneq \mathscr{M}_A(\mathrm{PT}) \subsetneq \mathscr{M}_A(\mathrm{LT})$.

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Definite languages [Brzozowski 1962] [Ginzburg 1966]

Notation: $\mathscr{B}(\mathscr{C})$ denotes the (finite) Boolean closure of $\mathscr{C}$.

Definite:

$$\mathrm{D} = \mathscr{B}\{A^*w \mid w \in A^*\}$$

Reverse definite:

$$\mathrm{RD} = \mathscr{B}\{wA^* \mid w \in A^*\}$$

Generalised definite:

$$\mathrm{GD} = \mathscr{B}\{uA^*v \mid u, v \in A^*\}$$

Remark: $\mathrm{D}, \mathrm{RD} \subsetneq \mathrm{GD} \subsetneq \mathrm{LT} = \mathscr{B}\{wA^*, A^*w, A^*wA^* \mid w \in A^*\}$

Measuring power of LT and GD

Proposition: $\mathscr{M}(\mathrm{LT}) = \mathscr{M}(\mathrm{GD})$.

Proof (sketch):

Because $\mathscr{M}$ is idempotent and preserves the closure property under Boolean operations, it is enough to show that $wA^*, A^*w, A^*wA^* \in \mathscr{M}(\mathrm{GD})$ for any word w.

But wA^*, A^*w is already in GD, we only have to show $A^*wA^* \in \mathscr{M}(\mathrm{GD})$.

Define $W_k = \{xwy \in A^* \mid x, y \in A^*, |x| \leq k\}$.

Each W_k is reverse definite, $W_k \subseteq A^*wA^*$, and satisfies $\lim_{k\to\infty} \delta_A(W_k) = 1$.

This means that W_k converges to A^*wA^* (from inner).

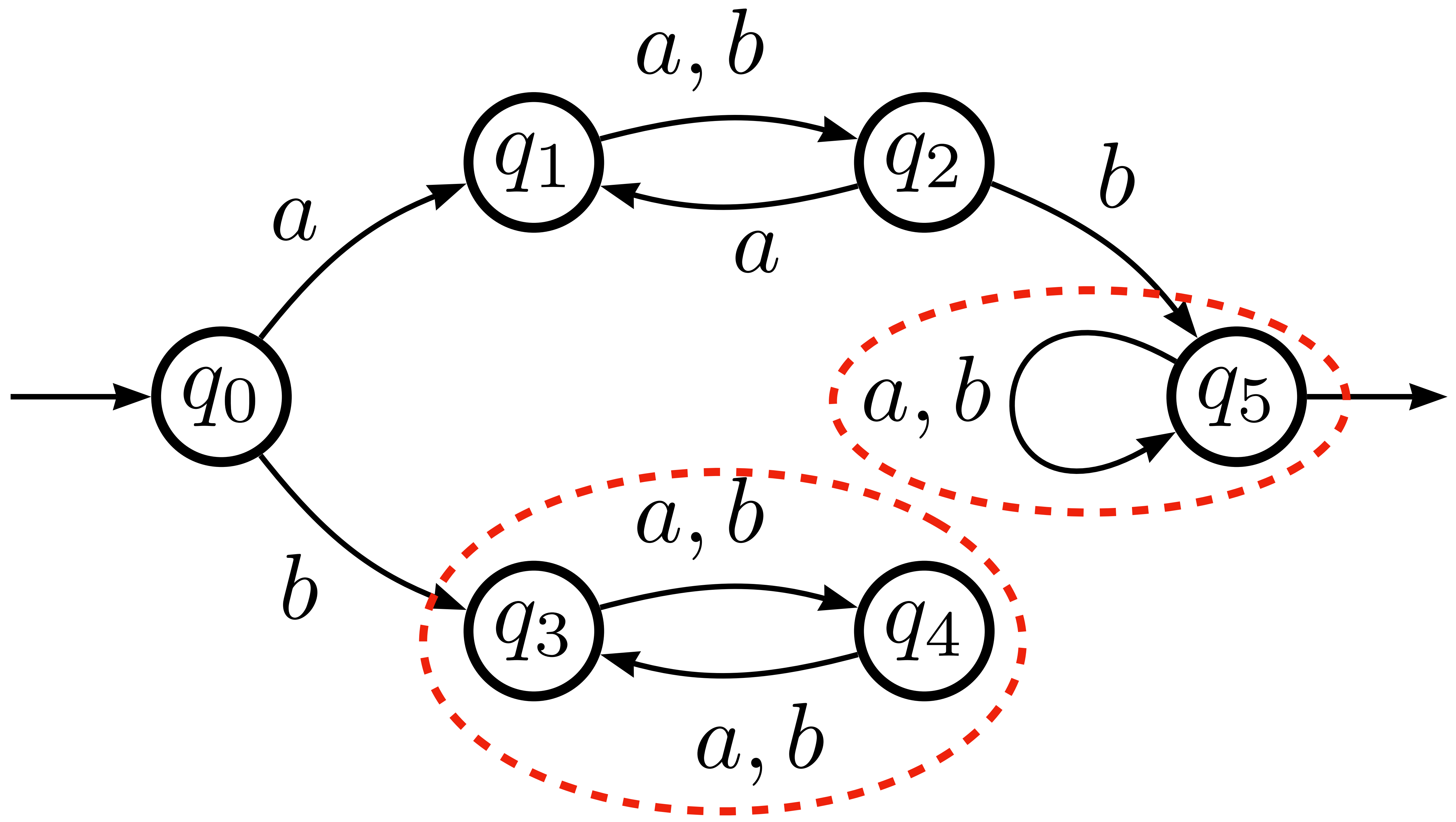
Sink component (a.k.a bottom strongly connected component)

Definition: Let $\mathscr{A} = (Q, \cdot, q_0, F)$ be a deterministic automaton.

A subset $S \subseteq Q$ is called sink if it satisfies following:

(1) S is strongly connected: $\forall p, q \in S \, \exists w \in A^* \, p \cdot w = q$

(2) S has no outgoing transition: $\forall p \in S \, \forall w \in A^* \, p \cdot w \in S$.



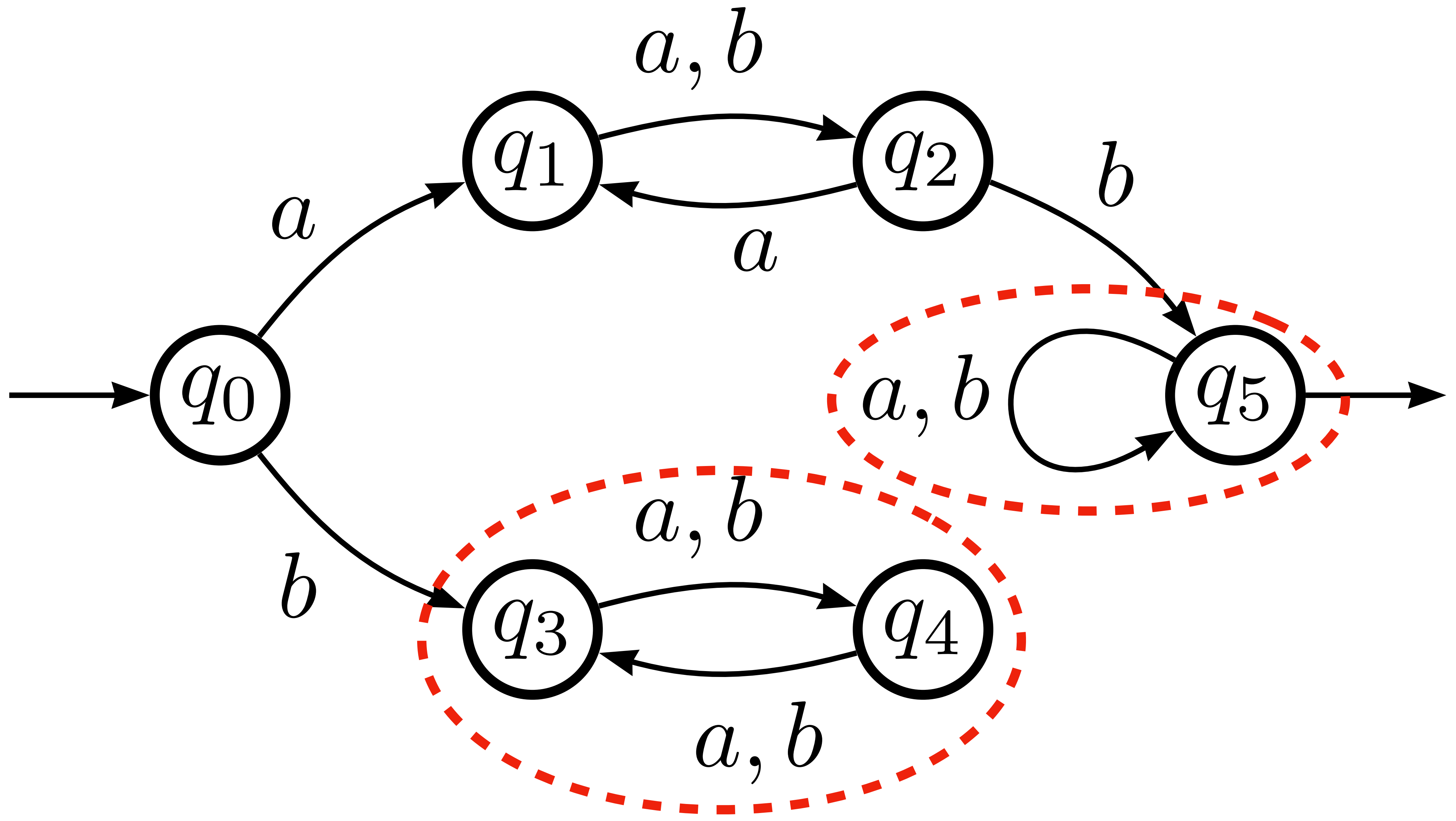
Sink components can be considered as a **minimal** SCC with respect to the reachability relation.

Characterisation of RD-measurable REGs

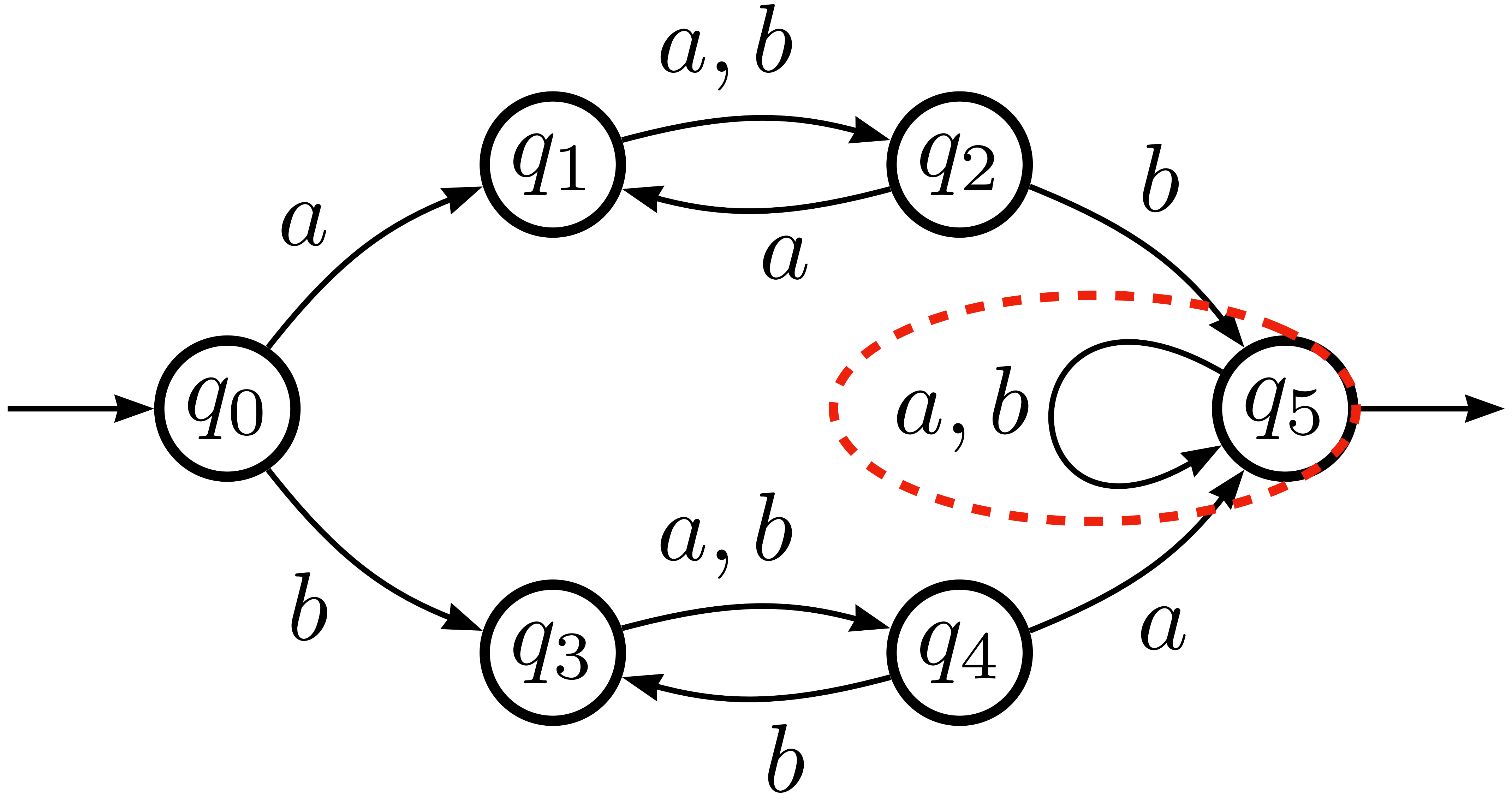
Theorem: Let $\mathscr{A} = (Q, \cdot, q_0, F)$ be a minimal deterministic automaton. TFAE:

(1) $L(\mathscr{A})$ is RD-measurable.

(2) Every sink component of $\mathscr{A}$ is a singleton (contains only one state).



RD-**im**measurable



RD-measurable

Characterisation of RD-measurable REGs

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(1) $L(\mathscr{A})$ is RD-measurable.

(2) Every sink component of $\mathscr{A}$ is a singleton (contains only one state).

Corollary: RD-measurability for minimal DFAs are decidable in linear time.

Corollary: D-measurability for DFAs are decidable.

Characterisation of GD-measurable REGs

Theorem: Let $\mathscr{A} = (Q, \cdot, q_0, F)$ be a deterministic automaton, and
$Q_1, \ldots, Q_k$ be its all sink components.
Define $P_i = \{w \in A^* \mid q_0 \cdot w \in Q_i\}$,
$S_i = \{w \in A^* \mid Q_i \cdot w \subseteq F\}$ and $S_i' = \{w \in A^* \mid Q_i \cdot w \cap F = \varnothing\}$.

Let $\quad M = \bigcup_{i=1}^{k} P_i S_i \quad$ and $\quad M' = \bigcup_{i=1}^{k} P_i S_i'$.

Then $L(\mathscr{A})$ is GD-measurable if and only if $\delta_A(M) + \delta_A(M') = 1$.

Characterisation of GD-measurable REGs

Intuition: M is a largest subset of $L(\mathscr{A})$ that can be represented as a (possibly infinite) union of languages of the form uA^*w .

That is, M is a largest GD-measurable subset of $L(\mathscr{A})$.

Also, M' is a largest GD-measurable subset of $\overline{L(\mathscr{A})}$.

Let $M = \bigcup_{i=1}^{k} P_i S_i$ and $M' = \bigcup_{i=1}^{k} P_i S_i'$.

Then $L(\mathscr{A})$ is GD-measurable if and only if $\delta_A(M) + \delta_A(M') = 1$.

This condition means $\delta_A(L(\mathscr{A})) = \delta_A(M)$ and $\delta_A(\overline{L(\mathscr{A})}) = \delta_A(M')$.

Characterisation of GD-measurable REGs

Theorem: Let $\mathscr{A} = (Q, \cdot, q_0, F)$ be a deterministic automaton, and $Q_1, \ldots, Q_k$ be its all sink components.

Define $P_i = \{w \in A^* \mid q_0 \cdot w \in Q_i\}$,

$S_i = \{w \in A^* \mid Q_i \cdot w \subseteq F\}$ and $S'_i = \{w \in A^* \mid Q_i \cdot w \cap F = \varnothing\}$.

Let $M = \bigcup_{i=1}^{k} P_i S_i$ and $M' = \bigcup_{i=1}^{k} P_i S'_i$.

Then $L(\mathscr{A})$ is GD-measurable if and only if $\delta_A(M) + \delta_A(M') = 1$.

Corollary: GD-measurability for DFAs is **decidable**.

(because the density of a regular language is computable and M, M' are regular by the construction)

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Summary

Theorem 1:
RD-measurability for DFAs is decidable in linear time.

Theorem 2:
The measuring power of GD and LT are equivalent, and
GD-measurability is decidable for DFAs (in PSPACE).

Progress: we (S., Y. Nakamura and Y. Yamaguchi) found that it is in PTIME.

Open problem

Is the measuring power of GD and SF (the class of all **star-free** languages) equivalent or not?

[S. DLT'22]: $\mathscr{M}_A(\mathrm{AT}) \subsetneq \mathscr{M}_A(\mathrm{PT}) \subsetneq \mathscr{M}_A(\mathrm{LT})$

$= \underline{\mathscr{M}_A(\mathrm{GD}) \subseteq \mathscr{M}_A(\mathrm{SF})}.$

Is this inclusion strict?

How much GD-measurability is weaker than regular measurability?

Open problem

[S. SOFSEM'21]: The set Q of all primitive words is regular ***im***measurable.

The proof uses non-trivial analysis of syntactic monoids of regular languages.

However, the GD-***im***measurability of Q is almost trivia:

Because uA^*v contains non-primitive word $uvuv$,
there is no infinite generalised definite subset of Q.

How much GD-measurability is weaker than regular measurability?

Application?

The decidable characterisation of GD-measurability gives us the following **approximation scheme**:

Input : an automaton $\mathscr{A}$ and an admissible error ratio $\epsilon > 0$.
Output: an automaton $\mathscr{B}$ (if exists) such that

(1) $L(\mathscr{A}) \subseteq L(\mathscr{B})$, (2) $L(\mathscr{B})$ is generalised definite,

and (3) $|\delta_A(L(\mathscr{A})) - \delta_A(L(\mathscr{A}))| \leq \epsilon$.

Can we apply this scheme to, say,

obtain an efficient regular expression matching algorithm?

(or other decision problems)?

Thanks!

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Some known properties of $\mathscr{C}$-measurability [S. DLT'21]

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- $\mathscr{M}_A(\mathscr{C})$ is closed under Boolean operations and left-and-right quotients if $\mathscr{C}$ is closed under Boolean operations and left-and-right quotients.

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- $\mathscr{M}_A$ is a closure operator:
  (extensive) $\mathscr{C} \subseteq \mathscr{M}_A(\mathscr{C})$ **(monotone)** $\mathscr{C} \subseteq \mathscr{D} \Rightarrow \mathscr{M}_A(\mathscr{C}) \subseteq \mathscr{M}_A(\mathscr{D})$
  (idempotent) $\mathscr{M}_A(\mathscr{M}_A(\mathscr{C})) = \mathscr{M}_A(\mathscr{C})$

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(1) S is strongly connected: $\forall p, q \in S \, \exists w \in A^* \, p \cdot w = q$

(2) S has no outgoing transition: $\forall p \in S \, \forall w \in A^* \, p \cdot w \in S$.

Fact: For any deterministic automaton $\mathscr{A} = (Q, \cdot, q_0, F)$,

the language $P = \{w \in A^* \mid q_0 \cdot w \text{ not in any sink component}\}$

has density zero.