Coequalisers in the category of basic pairs

Hajime Ishihara
joint work with Takako Nemoto

School of Information Science
Japan Advanced Institute of Science and Technology
(JAIST)
Nomi, Ishikawa 923-1292, Japan

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自己紹介

氏名：石原　哉

略歴：

1980	東京工業大学情報科学科卒業
1987	東京工業大学大学院情報科学専攻修士課程修了
1988	広島大学総合科学部
1990	理学博士（東京工業大学）
1992	北陸先端科学技術大学院大学

研究：構成的数学、数理論理学、計算論

A history of constructivism

- History
  - Arithmetization of mathematics (Kronecker, 1887)
  - Three kinds of intuition (Poincaré, 1905)
  - French semi-intuitionism (Borel, 1914)
  - Intuitionism (Brouwer, 1914)
  - Predicativity (Weyl, 1918)
  - Finitism (Skolem, 1923; Hilbert-Bernays, 1934)
  - Constructive recursive mathematics (Markov, 1954)
  - Constructive mathematics (Bishop, 1967)
- Logic
  - Intuitionistic logic (Heyting, 1934; Kolmogorov, 1932)

Language

We use the standard language of (many-sorted) first-order predicate logic based on

- primitive logical operators $\wedge, \vee, \rightarrow, \bot, \forall, \exists$.

We introduce the abbreviations

- $\neg A \equiv A \rightarrow \bot$;
- $A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$.

The BHK interpretation

The Brouwer-Heyting-Kolmogorov (BHK) interpretation of the logical operators is the following.

- A proof of $A \wedge B$ is given by presenting a proof of A and a proof of B.
- A proof of $A \vee B$ is given by presenting either a proof of A or a proof of B.
- A proof of $A \to B$ is a construction which transforms any proof of A into a proof of B.
- Absurdity $\bot$ has no proof.
- A proof of $\forall x A(x)$ is a construction which transforms any t into a proof of $A(t)$.
- A proof of $\exists x A(x)$ is given by presenting a t and a proof of $A(t)$.

The BHK interpretation

- A proof of $\forall x \exists y A(x, y)$ is a construction which transforms any t into a proof of $\exists y A(t, y)$;
- A proof of $\exists y A(t, y)$ is given by presenting an s and a proof of $A(t, s)$.

Therefore

- a proof of $\forall x \exists y A(x, y)$ is a construction which transforms any t into s and a proof of $A(t, s)$.

Remark 1

- A proof of $\neg(\neg A \wedge \neg B)$ is not a proof of $A \vee B$.
- A proof of $\neg \forall x \neg A(x)$ is not a n proof of $\exists x A(x)$.

Natural Deduction System

We shall use $\mathcal{D}$, possibly with a subscript, for arbitrary deduction.

We write

$$\begin{array}{c}\Gamma \\ \mathcal{D} \\ A\end{array}$$

to indicate that $\mathcal{D}$ is deduction with conclusion A and assumptions Γ.

Deduction (Basis)

For each formula A,

$$A$$

is a deduction with conclusion A and assumptions $\{A\}$.

Deduction (Induction step, $\to$I)

If

$$\begin{array}{c}\Gamma\\ \mathcal{D}\\ B\end{array}$$

is a deduction, then

$$\dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ B\end{array}}{A \to B}\ \to\mathrm{I}$$

is a deduction with conclusion $A \to B$ and assumptions $\Gamma \setminus \{A\}$.
We write

$$\dfrac{\begin{array}{c}[A]\\ \mathcal{D}\\ B\end{array}}{A \to B}\ \to\mathrm{I}$$

Deduction (Induction step, $\rightarrow\mathrm{E}$)

If

$$\begin{array}{ccc} \Gamma_1 & & \Gamma_2 \\ \mathcal{D}_1 & & \mathcal{D}_2 \\ A \rightarrow B & & A \end{array}$$

are deductions, then

$$\dfrac{\begin{array}{cc} \Gamma_1 & \Gamma_2 \\ \mathcal{D}_1 & \mathcal{D}_2 \\ A \rightarrow B & A \end{array}}{B} \rightarrow\mathrm{E}$$

is a deduction with conclusion B and assumptions $\Gamma_1 \cup \Gamma_2$.

Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\neg\neg A] \qquad \dfrac{\dfrac{[\neg\neg(A \to B)] \qquad \dfrac{\dfrac{[\neg B] \qquad \dfrac{[A \to B] \quad [A]}{B}\,{\to}\mathrm{E}}{\bot}\,{\to}\mathrm{E}}{\neg(A \to B)}\,{\to}\mathrm{I}}{\bot}\,{\to}\mathrm{E}}{\neg A}\,{\to}\mathrm{I}}{\bot}\,{\to}\mathrm{E}}{\neg\neg B}\,{\to}\mathrm{I}}{\neg\neg A \to \neg\neg B}\,{\to}\mathrm{I}}{\neg\neg(A \to B) \to (\neg\neg A \to \neg\neg B)}\,{\to}\mathrm{I}
$$

Minimal logic

$$\dfrac{\begin{matrix}[A]\\ \mathcal{D}\\ B\end{matrix}}{A \to B}\ \to\mathrm{I} \qquad \dfrac{\begin{matrix}\mathcal{D}_1\\ A \to B\end{matrix} \quad \begin{matrix}\mathcal{D}_2\\ A\end{matrix}}{B}\ \to\mathrm{E}$$

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ A\end{matrix} \quad \begin{matrix}\mathcal{D}_2\\ B\end{matrix}}{A \wedge B}\ \wedge\mathrm{I} \qquad \dfrac{\begin{matrix}\mathcal{D}\\ A \wedge B\end{matrix}}{A}\ \wedge\mathrm{E}_r \qquad \dfrac{\begin{matrix}\mathcal{D}\\ A \wedge B\end{matrix}}{B}\ \wedge\mathrm{E}_l$$

$$\dfrac{\begin{matrix}\mathcal{D}\\ A\end{matrix}}{A \vee B}\ \vee\mathrm{I}_r \qquad \dfrac{\begin{matrix}\mathcal{D}\\ B\end{matrix}}{A \vee B}\ \vee\mathrm{I}_l \qquad \dfrac{\begin{matrix}\mathcal{D}_1\\ A \vee B\end{matrix} \quad \begin{matrix}[A]\\ \mathcal{D}_2\\ C\end{matrix} \quad \begin{matrix}[B]\\ \mathcal{D}_3\\ C\end{matrix}}{C}\ \vee\mathrm{E}$$

Minimal logic

$$\dfrac{\begin{matrix}\mathcal{D}\\ A\end{matrix}}{\forall y A[x/y]}\ \forall\mathrm{I} \qquad \dfrac{\begin{matrix}\mathcal{D}\\ \forall x A\end{matrix}}{A[x/t]}\ \forall\mathrm{E}$$

$$\dfrac{\begin{matrix}\mathcal{D}\\ A[x/t]\end{matrix}}{\exists x A}\ \exists\mathrm{I} \qquad \dfrac{\begin{matrix}\mathcal{D}_1\\ \exists y A[x/y]\end{matrix} \quad \begin{matrix}[A]\\ \mathcal{D}_2\\ C\end{matrix}}{C}\ \exists\mathrm{E}$$

- In $\forall\mathrm{E}$ and $\exists\mathrm{I}$, t must be free for x in A.
- In $\forall\mathrm{I}$, $\mathcal{D}$ must not contain assumptions containing x free, and $y \equiv x$ or $y \notin \mathrm{FV}(A)$.
- In $\exists\mathrm{E}$, $\mathcal{D}_2$ must not contain assumptions containing x free except A, $x \notin \mathrm{FV}(C)$, and $y \equiv x$ or $y \notin \mathrm{FV}(A)$.

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[(A \to B) \land (A \to C)]}{A \to B}\,\land\mathrm{E}_r \qquad [A]}{B}\,\to\mathrm{E} \qquad \dfrac{\dfrac{[(A \to B) \land (A \to C)]}{A \to C}\,\land\mathrm{E}_l \qquad [A]}{C}\,\to\mathrm{E}}{B \land C}\,\land\mathrm{I}}{A \to B \land C}\,\to\mathrm{I}}{(A \to B) \land (A \to C) \to (A \to B \land C)}\,\to\mathrm{I}$$

Intuitionistic logic

Intuitionistic logic is obtained from minimal logic by adding the intuitionistic absurdity rule (ex falso quodlibet).

If

$$\begin{array}{c}\Gamma\\ \mathcal{D}\\ \bot\end{array}$$

is a deduction, then

$$\begin{array}{c}\Gamma\\ \mathcal{D}\\ \dfrac{\bot}{A}\ \bot_i\end{array}$$

is a deduction with conclusion A and assumptions Γ.

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg\neg A \to \neg\neg B] \qquad \dfrac{\dfrac{[\neg(A\to B)] \qquad \dfrac{\dfrac{\dfrac{[\neg A] \quad [A]}{\bot}\,{\to}\mathrm{E}}{B}\,\color{red}{\bot_i}}{A\to B}\,{\to}\mathrm{I}}{\bot}\,{\to}\mathrm{E}}{\neg\neg A}\,{\to}\mathrm{I}}{\neg\neg B}\,{\to}\mathrm{E} \qquad \dfrac{\dfrac{[\neg(A\to B)] \quad \dfrac{[B]}{A\to B}\,{\to}\mathrm{I}}{\bot}\,{\to}\mathrm{E}}{\neg B}\,{\to}\mathrm{I}}{\bot}}{\neg\neg(A\to B)}\,{\to}\mathrm{I}}{(\neg\neg A\to\neg\neg B)\to\neg\neg(A\to B)}\,{\to}\mathrm{I}$$

Example

$$\dfrac{\dfrac{\dfrac{[A \lor B] \qquad \dfrac{\dfrac{[\neg A] \quad [A]}{\bot}\,{\to}\mathrm{E}}{B}\,{\color{red}\bot_i} \qquad [B]}{B}\,{\lor}\mathrm{E}}{\neg A \to B}\,{\to}\mathrm{I}}{A \lor B \to (\neg A \to B)}\,{\to}\mathrm{I}$$

Classical logic

Classical logic is obtained from intuitionistic logic by strengthening the absurdity rule to the classical absurdity rule (reductio ad absurdum).

If

$$\begin{array}{c}\Gamma\\ \mathcal{D}\\ \bot\end{array}$$

is a deduction, then

$$\dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ \bot\end{array}}{A}\ \bot_c$$

is a deduction with conclusion A and assumption $\Gamma \setminus \{\neg A\}$.

Example (classical logic)

The double negation elimination (DNE):

$$\dfrac{\dfrac{\dfrac{[\neg\neg A] \quad [\neg A]}{\bot}\ \to\mathrm{E}}{A}\ \bot_c}{\neg\neg A \to A}\ \to\mathrm{I}$$

Example (classical logic)

The principle of excluded middle (PEM):

$$\dfrac{\dfrac{[\neg(A\vee\neg A)] \qquad \dfrac{\dfrac{\dfrac{[\neg(A\vee\neg A)] \qquad \dfrac{[A]}{A\vee\neg A}\,\vee\mathrm{I}_r}{\bot}\,\to\mathrm{E}}{\neg A}\,\to\mathrm{I}}{A\vee\neg A}\,\vee\mathrm{I}_l}{\bot}\,\to\mathrm{E}}{A\vee\neg A}\,\bot_c$$

Example (classical logic)

De Morgan's law (DML):

$$
\dfrac{\dfrac{\dfrac{[\neg(\neg A \vee \neg B)] \qquad \dfrac{\dfrac{\dfrac{[\neg(\neg A \vee \neg B)] \qquad \dfrac{\dfrac{\dfrac{[\neg(A \wedge B)] \qquad \dfrac{[A] \quad [B]}{A \wedge B}\,\wedge\mathrm{I}}{\bot}\,\to\mathrm{E}}{\neg A}\,\to\mathrm{I}}{\neg A \vee \neg B}\,\vee\mathrm{I}_r}{\bot}\,\to\mathrm{E}}{\neg B}\,\to\mathrm{I}}{\neg A \vee \neg B}\,\vee\mathrm{I}_l}{\bot}\,\to\mathrm{E}}{\neg A \vee \neg B}\,\bot_c}{\neg(A \wedge B) \to \neg A \vee \neg B}\,\to\mathrm{I}
$$

RAA vs $\rightarrow\mathrm{I}$

$\bot_c$: deriving A by deducing absurdity ($\bot$) from $\neg A$.

$$\dfrac{\begin{array}{c}[\neg A]\\ \mathcal{D}\\ \bot\end{array}}{A}\ \bot_c$$

$\rightarrow\mathrm{I}$: deriving $\neg A$ by deducing absurdity ($\bot$) from A.

$$\dfrac{\begin{array}{c}[A]\\ \mathcal{D}\\ \bot\end{array}}{\neg A}\ \rightarrow\mathrm{I}$$

A short history

- Aczel (2006) introduced the notion of a set-generated class for dcpos using some terminology from domain theory.
- van den Berg (2013) introduced the principle NID on non-deterministic inductive definitions and set-generated classes in the constructive Zermelo-Frankel set theory **CZF**.
- Aczel et al. (2015) characterized set-generated classes using generalized geometric theories and a set generation axiom SGA in **CZF**.
- I-Kawai (2015) constructed coequalisers in the category of basic pairs in the extension of **CZF** with SGA.
- I-Nemoto (2016) introduced another NID principle, called nullary NID, and proved that nullary NID is equivalent to Fullness in a subsystem **ECST** of **CZF**.

The elementary constructive set theory

The language of a constructive set theory contains variables for sets and the binary predicates $=$ and $\in$. The axioms and rules are those of intuitionistic predicate logic with equality. In addition, **ECST** has the following set theoretic axioms:

Extensionality: $\forall a \forall b[\forall x(x \in a \leftrightarrow x \in b) \rightarrow a = b]$.

Pairing: $\forall a \forall b \exists c \forall x(x \in c \leftrightarrow x = a \vee x = b)$.

Union: $\forall a \exists b \forall x[x \in b \leftrightarrow \exists y \in a(x \in y)]$.

Restricted Separation:

$$\forall a \exists b \forall x(x \in b \leftrightarrow x \in a \wedge \varphi(x))$$

for every *restricted* formula $\varphi(x)$. Here a formula $\varphi(x)$ is restricted, or Δ_0, if all the quantifiers occurring in it are bounded, i.e. of the form $\forall x \in c$ or $\exists x \in c$.

The elementary constructive set theory

Replacement:

$$\forall a[\forall x \in a \exists! y \varphi(x,y) \rightarrow \exists b \forall y (y \in b \leftrightarrow \exists x \in a \varphi(x,y))]$$

for every formula $\varphi(x,y)$.

Strong Infinity:

$$\exists a[0 \in a \wedge \forall x(x \in a \rightarrow x+1 \in a) \\ \wedge \forall y(0 \in y \wedge \forall x(x \in y \rightarrow x+1 \in y) \rightarrow a \subseteq y)],$$

where $x+1$ is $x \cup \{x\}$, and 0 is the empty set $\emptyset$.

The elementary constructive set theory

- Using Replacement and Union, the cartesian product $a \times b$ of sets a and b consisting of the ordered pairs $(x, y) = \{\{x\}, \{x, y\}\}$ with $x \in a$ and $y \in b$ can be introduced in **ECST**.
- A relation r between a and b is a subset of $a \times b$. A relation $r \subseteq a \times b$ is total (or is a multivalued function) if for every $x \in a$ there exists $y \in b$ such that $(x, y) \in r$.
- A function from a to b is a total relation $f \subseteq a \times b$ such that for every $x \in a$ there is exactly one $y \in b$ with $(x, y) \in f$.

The elementary constructive set theory

The class of total relations between a and b is denoted by $\mathrm{mv}(a, b)$:

$$r \in \mathrm{mv}(a, b) \Leftrightarrow r \subseteq a \times b \wedge \forall x \in a \exists y \in b((x, y) \in r).$$

The class of functions from a to b is denoted by b^a:

$$f \in b^a \Leftrightarrow f \in \mathrm{mv}(a, b) \\ \wedge \forall x \in a \forall y, z \in b((x, y) \in f \wedge (x, z) \in f \rightarrow y = z).$$

The constructive set theory **CZF**

The constructive set theory **CZF** is obtained from **ECST** by replacing Replacement by

Strong Collection:

$$\forall a[\forall x \in a \exists y \varphi(x,y) \rightarrow \exists b(\forall x \in a \exists y \in b \varphi(x,y) \\ \land \forall y \in b \exists x \in a \varphi(x,y))]$$

for every formula $\varphi(x,y)$,

The constructive set theory **CZF**

and adding

Subset Collection:

$$\begin{aligned}\forall a \forall b \exists c \forall u [\forall x \in a \exists y \in b \varphi(x, y, u) \rightarrow \\ \exists d \in c (\forall x \in a \exists y \in d \varphi(x, y, u) \\ \wedge \forall y \in d \exists x \in a \varphi(x, y, u))]\end{aligned}$$

for every formula $\varphi(x, y, u)$, and

$\in$-Induction:

$$\forall a (\forall x \in a \varphi(x) \rightarrow \varphi(a)) \rightarrow \forall a \varphi(a),$$

for every formula $\varphi(a)$.

The constructive set theory **CZF**

- In **ECST**, Subset Collection implies

  Fullness:

$$\forall a \forall b \exists c (c \subseteq \mathrm{mv}(a, b) \\ \wedge \forall r \in \mathrm{mv}(a, b) \exists s \in c (s \subseteq r)),$$

  and Fullness and Strong Collection imply Subset Collection.
- The notable consequence of Fullness is that b^a forms a set:
  Exponentiation: $\forall a \forall b \exists c \forall f (f \in c \leftrightarrow f \in b^a)$.
- For a set S, we write $\mathrm{Pow}(S)$ for the power class of S which is not a set in **ECST** nor in **CZF**:

$$a \in \mathrm{Pow}(S) \Leftrightarrow a \subseteq S.$$

Set-generated classes

Definition 2

Let S be a set, and let X be a subclass of $\mathrm{Pow}(S)$. Then X is set-generated if there exists a subset G, called a generating set, of X such that

$$\forall \alpha \in X \forall x \in \alpha \exists \beta \in G (x \in \beta \subseteq \alpha).$$

Remark 3

The power class $\mathrm{Pow}(S)$ of a set S is set-generated with a generating set

$$\{\{x\} \mid x \in S\}.$$

Rules

Definition 4

Let S be a set. Then a rule on S is a pair (a, b) of subsets a and b of S. A rule is called

- nullary if a is empty;
- elementary if a is a singleton;
- finitary if a is finitely enumerable.

A subset α of S is closed under a rule (a, b) if

$$a \subseteq \alpha \to b \between \alpha.$$

For a set R of rules on S, we call a subset α of S R-closed if it is closed under each rule in R.

Remark 5

Note that if a rule is nullary or elementary, then it is finitary.

NID principles

Definition 6

Let NID denote the principles that

- for each set S and set R of rules on S, the class of R-closed subsets of S is set-generated.

The principles obtained by restricting R in NID to a set of nullary, elementary and finitary rules are denoted by NID_0, NID_1 and $\mathrm{NID}_{<\omega}$, respectively.

Remark 7

Note that $\mathrm{NID}_{<\omega}$ implies NID_0 and NID_1.

The nullary NID

Theorem 8 (I-Nemoto 2015)

The following are equivalent over **ECST**.

1. NID_0.
2. *Fullness.*

Proposition 9 (I-Nemoto 2015)

NID_1 *implies* NID_0.

Remark 10

$$\mathrm{NID}_0 \longleftarrow \mathrm{NID}_1 \longleftarrow \mathrm{NID}_{<\omega}$$

Basic pairs

Definition 11

A basic pair is a triple $(X, \Vdash, S)$ of sets X and S, and a relation $\Vdash$ between X and S.

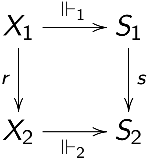
Relation pairs

Definition 12

A relation pair between basic pairs $\mathcal{X}_1 = (X_1, \Vdash_1, S_1)$ and $\mathcal{X}_2 = (X_2, \Vdash_2, S_2)$ is a pair (r, s) of relations $r \subseteq X_1 \times X_2$ and $s \subseteq S_1 \times S_2$ such that

$$\Vdash_2 \circ\, r = s \circ \Vdash_1,$$

that is, the following diagram commute.



Relation pairs

Definition 13

Two relation pairs (r_1, s_1) and (r_2, s_2) between basic pairs $\mathcal{X}_1$ and $\mathcal{X}_2$ are equivalent, denoted by $(r_1, s_1) \sim (r_2, s_2)$, if

$$\Vdash_2 \circ\, r_1 = \Vdash_2 \circ\, r_2,$$

or equivalently $s_1 \circ \Vdash_1 = s_2 \circ \Vdash_1$.

The category of basic pairs

Notation 14

For a basic pair $(X, \Vdash, S)$, we write

$$\Diamond D = \Vdash(D) \quad \text{and} \quad \mathrm{ext}\, U = \Vdash^{-1}(U)$$

for $D \in \mathrm{Pow}(X)$ and $U \in \mathrm{Pow}(S)$.

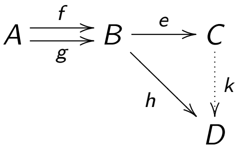
Proposition 15

Basic pairs and relation pairs form a category **BP**.

Coequalisers

Definition 16

A coequaliser of a parallel pair $A \underset{g}{\overset{f}{\rightrightarrows}} B$ in a category $\mathbf{C}$ is a pair of an object C and a morphism $B \xrightarrow{e} C$ such that $e \circ f = e \circ g$, and it satisfies a universal property in the sense that for any morphism $B \xrightarrow{h} D$ with $h \circ f = h \circ g$, there exists a unique morphism $C \xrightarrow{k} D$ for which the following diagram commutes.



Coequalisers

Proposition 17 (I-Kawai 2015)

Let $\mathcal{X}_1 \overset{(r_1,s_1)}{\underset{(r_2,s_2)}{\rightrightarrows}} \mathcal{X}_2$ *be a parallel pair of relation pairs in* **BP**. *If a subclass*

$$Q = \{U \in \mathrm{Pow}(S_2) \mid \mathrm{ext}_1\, s_1^{-1}(U) = \mathrm{ext}_1\, s_2^{-1}(U)\}$$

of $\mathrm{Pow}(S_2)$ *is set-generated, then the parallel pair has a coequaliser.*

A NID principle

Definition 18

Let S be a set. Then a subset α of S is biclosed under a rule (a, b) if

$$a \between \alpha \leftrightarrow b \between \alpha.$$

For a set R of rules on S, we call a subset α of S R-biclosed if it is biclosed under each rule in R.

Definition 19

Let $\mathrm{NID}_{\mathrm{bi}}$ denotes the principles that

- for each set S and set R of rules on S, the class of R-biclosed subsets of S is set-generated.

A NID principle

Proposition 20

- NID_1 *implies* $\mathrm{NID}_{\mathrm{bi}}$.
- $\mathrm{NID}_{\mathrm{bi}}$ *implies* NID_0.

Remark 21

$$\mathrm{NID}_0 \longleftarrow \mathrm{NID}_{\mathrm{bi}} \longleftarrow \mathrm{NID}_1 \longleftarrow \mathrm{NID}_{<\omega}$$

BP has coequalisers

Theorem 22

The following are equivalent over **ECST***.*

1. $\mathrm{NID}_{\mathrm{bi}}$.
2. **BP** *has coequalisers.*

Remark 23

Since **BP** has small coproducts, in the presence of $\mathrm{NID}_{\mathrm{bi}}$, the category **BP** is cocomplete.

Work in progress

Definition 24

A rule (a, b) on a set S is called n-ary if there exists a surjection $n \to a$.

Remark 25

Note that if a rule is $n+1$-ary, then it is $n+2$-ary.

Definition 26

The principles obtained by restricting R in NID to a set of n-ary rules is denoted by NID_n.

Work in progress

Proposition 27

NID_2 *implies* $\mathrm{NID}_{<\omega}$.

Remark 28

$$\mathrm{NID}_0 \longleftarrow \mathrm{NID}_{\mathrm{bi}} \longleftarrow \mathrm{NID}_1 \longleftarrow \mathrm{NID}_2 \longleftrightarrow \cdots \longleftrightarrow \mathrm{NID}_{<\omega}$$

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