MacNeille completion and Buchholz' Omega rule

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Introduction:

犬の口にはゴムパッキンがついている
（佐々木倫子『動物のお医者さん』より）

A similarity

Buchholz' Ω-rule (1981)

$$\frac{\{\ \Delta \Rightarrow \Pi^*\ \}_{\Delta \Rightarrow^{\mathbf{LI}}_{Y} \varphi^*(Y)}}{\forall X.\varphi(X) \Rightarrow \Pi}$$

where Δ is 1st order and $\forall X.\varphi(X),\ \Pi$ is 2nd order,

is similar to

a characteristic property of MacNeille completion

$\mathbf{A} \subseteq \overline{\mathbf{A}}$:

$$\frac{\{a \leq y\}_{a \leq x}}{x \leq y}$$

where $a \in \mathbf{A}$ and $x, y \in \overline{\mathbf{A}}$.

Cut elimination proofs for higher order logics/arithmetic

Syntactic cut elimination

1. Ordinal assignment
2. Ω-rule technique (Buchholz, Aehlig, Mints, Akiyoshi, . . .). Works only for fragments of higher order logics/arithmetic.

Semantic cut elimination

1. Semi-valuation (Schütte, Takahashi, Prawitz). 3-valued semantics (Girard 76). Employs RAA and WKL. Destroys the proof structure.
2. MacNeille completion and reducibility candidates (Maehara 91, Okada 96, after Girard 71). Fully constructive. Extends to strong normalization.

Cut elimination proofs for higher order logics/arithmetic

Target system	Fragments	Full higher-order logics
Algebraic proof	???	MacNeille completion + reducibility candidates
Syntactic proof	Ω-rule	Takeuti's Conjecture

In this talk we fill in the ??? slot by introducing the concept of Ω-valuation. The target systems are parameter-free 2nd order intuitionistic logics.

Cut elimination proofs for higher order logics/arithmetic

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In this talk we fill in the ??? slot by introducing the concept of Ω-valuation. The target systems are parameter-free 2nd order intuitionistic logics.

Notice: It is mostly a reworking of known results (especially those of Klaus Aehlig). Our purpose is just to provide an algebraic perspective on them.

Outline

- MacNeille completion
- Parameter-free 2nd order intuitionistic logics
- Ω-rule technique (syntactic)
- Ω-valuation technique (semantic)
- For the lambda calculus audience
- For the nonclassical logics audience

MacNeille completion

石器時代より前のおはなし

MacNeille completion

$\mathbf{A}$: a lattice.
A completion of $\mathbf{A}$ is an embedding $e : \mathbf{A} \longrightarrow \mathbf{B}$ into a complete lattice $\mathbf{B}$ (we often assume $\mathbf{A} \subseteq \mathbf{B}$).
Examples:

- $\mathbb{Q} \subseteq \mathbb{R} \cup \{\pm\infty\}$
- $e : \mathbf{A} \longrightarrow \wp(\mathrm{uf}(\mathbf{A}))$ ($\mathbf{A}$: Boolean algebra)

$\mathbf{A} \subseteq \mathbf{B}$ is a MacNeille completion if for any $x \in \mathbf{B}$,

$$x = \bigwedge\{a \in \mathbf{A} : x \leq a\} = \bigvee\{a \in \mathbf{A} : a \leq x\}.$$

Theorem (Banachewski 56, Schmidt 56)

Every lattice $\mathbf{A}$ has a unique MacNeille completion $\overline{\mathbf{A}}$.
MacNeille completion is regular, i.e., preserves $\bigwedge$ and $\bigvee$ that already exist in $\mathbf{A}$.

MacNeille completion

- $\mathbb{Q} \subseteq \mathbb{R} \cup \{\pm\infty\}$ is MacNeille, since

$$x = \inf\{a \in \mathbb{Q} : x \leq a\} = \sup\{a \in \mathbb{Q} : a \leq x\}$$

for any $x \in \mathbb{R}$. It is regular, e.g.,

$$0 = \lim_{n\to\infty} \frac{1}{n} \text{ (in } \mathbb{Q}) = \lim_{n\to\infty} \frac{1}{n} \text{ (in } \mathbb{R}).$$

- $e : \mathbf{A} \longrightarrow \wp(\mathrm{uf}(\mathbf{A}))$ is not regular, hence not MacNeille (actually a canonical extension).
- $f : \mathbf{B} \longrightarrow \mathrm{UpSet}(\mathrm{PPF}(\mathbf{B}))$ is not regular ($\mathbf{B}$: Heyting algebra)

MacNeille completion: its limitation

$\mathcal{DL}$: the class of distributive lattices.
$\mathcal{HA}$: the class of Heyting algebras.
$\mathcal{BA}$: the class of Boolean algebras.

Theorem

- $\mathcal{DL}$ is not closed under MacNeille (Funayama 44).
- $\mathcal{HA}$ and $\mathcal{BA}$ are closed under MacNeille completions.
- These are the only nontrivial subvarieties of $\mathcal{HA}$ closed under MacNeille (Harding-Bezhanishvili 04).

Conservative extension by MacNeille completion does not work for proper intermediate logics.

MacNeille completion: link to Ω-rule

Fact

A completion $\mathbf{A} \subseteq \mathbf{B}$ is MacNeille iff the inferences below are valid:

$$\frac{\{a \leq y\}_{a \leq x}}{x \leq y} \qquad \frac{\{x \leq a\}_{y \leq a}}{x \leq y}$$

where x, y range over $\mathbf{B}$ and a over $\mathbf{A}$.

"If $a \leq x$ implies $a \leq y$ for any $a \in \mathbf{A}$, then $x \leq y$."

This looks similar to the Ω-rule.

Parameter-free 2nd order intuitionistic logic

近年、若者の×××離れが著しい

Starter: full 2nd order logic

$\mathbf{G^1LI}$: sequent calculus for 2nd order intuitionistic logic with full comprehension

$$\frac{\varphi(\lambda x.\psi), \Gamma \Rightarrow \Pi}{\forall X.\varphi(X), \Gamma \Rightarrow \Pi} \qquad \frac{\Gamma \Rightarrow_Y \varphi(Y)}{\Gamma \vdash \forall X.\varphi(X)}$$

where

- $\Gamma \Rightarrow_Y \varphi(Y)$ means $Y \notin FV(\Gamma)$ (eigenvariable).
- $\varphi(\lambda x.\psi)$ obtained by replacing $t \in X \mapsto \psi(t)$.

Theorem (cf. Takeuti 53)

For any Σ^0_1 sentence φ,

$$\mathbf{Z}_2 \vdash \varphi \quad \Longrightarrow \quad \mathbf{G}^1\mathbf{LI} \vdash \xi \to \varphi$$

for some true Π^0_1 sentence ξ.
Cut elimination for $\mathbf{G}^1\mathbf{LI}$ implies 1-consistency of $\mathbf{Z}_2$, i.e., provable Σ^0_1-sentences are true.

Proof: By relativization $\varphi \mapsto \varphi^{\mathbf{N}}$.

$$\begin{array}{lcl} \mathbf{N}(t) & := & \forall X.[\forall x(x \in X \to x+1 \in X) \wedge 0 \in X \to t \in X] \\ (\forall x.\varphi)^{\mathbf{N}} & := & \forall x.\mathbf{N}(x) \to \varphi^{\mathbf{N}} \\ (\exists x.\varphi)^{\mathbf{N}} & := & \exists x.\mathbf{N}(x) \wedge \varphi^{\mathbf{N}} \end{array}$$

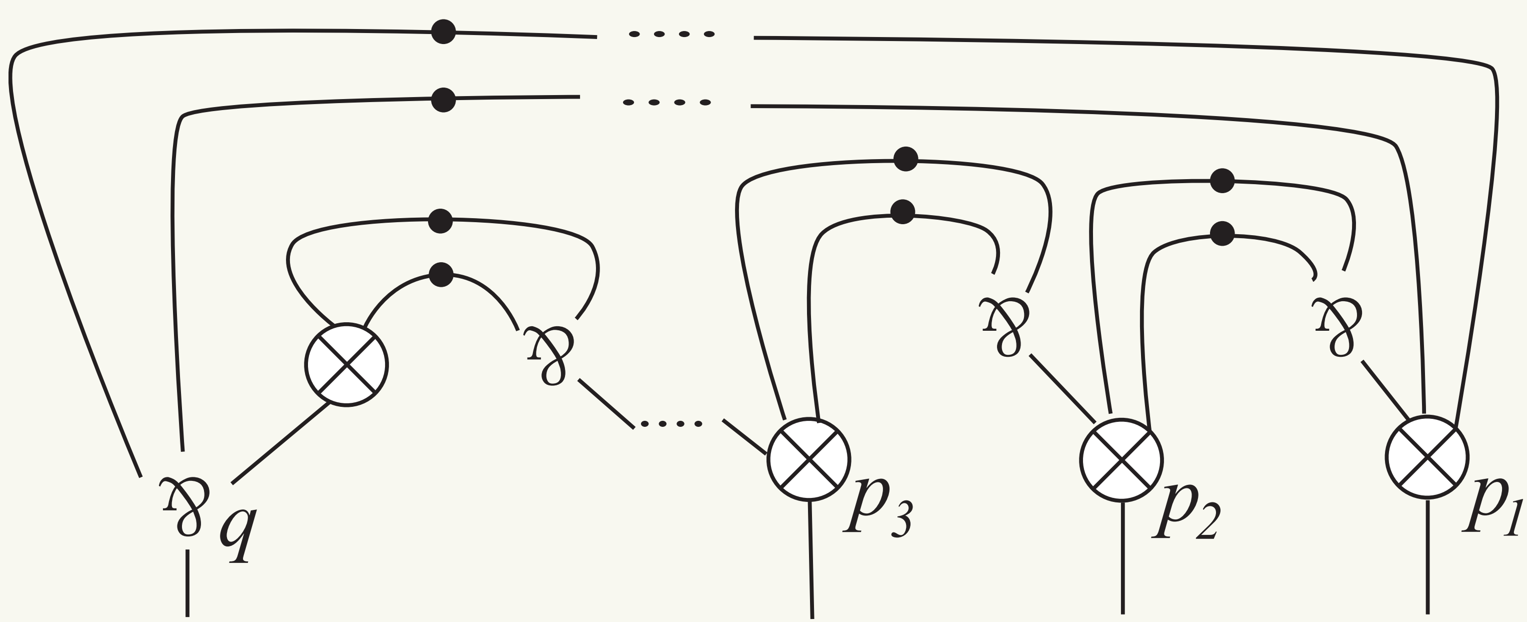
線形論理の「基礎論離れ」の系譜

- 1953 年：竹内、高階算術の無矛盾性を高階述語論理のカット除去に還元
- 1965 年：Prawitz、一般証明論の提唱
- 1971 年：Girard、高階命題論理の強正規化定理
- 1986 年：Girard、線形論理と証明ネットの提唱

証明ネットの理論が完全にうまくいくのは乗法的部分のみ：

$$\alpha \qquad \alpha^{\perp} \qquad A \otimes B \qquad A \wp B$$

乗法的部分に制限するなら論理式なんていらない。大事なのは証明ネットのグラフ構造のみ。



Parameter-free fragments of 2nd order intuitionistic logic

Tm: the set of 1st order terms

$X, Y, Z, \ldots$: 2nd order variables

Fm : the formulas of 1st-order intuitionistic logic

$$\varphi, \psi ::= p(\bar{t}) \mid t \in X \mid \bot \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \to \psi \mid \forall x.\varphi \mid \exists x.\varphi$$

FM_0:

$$\varphi ::= p(\bar{t}) \mid t \in X \mid \cdots \mid \forall X.\psi \mid \exists X.\psi$$

where $\psi \in \mathrm{Fm}$ doesn't contain 2nd order variables except X.

$\mathrm{FM}_1, \mathrm{FM}_2, \mathrm{FM}_3, \ldots$

If φ arithmetical, $\varphi^{\mathbf{N}} \in \mathrm{FM}_0$.

$\mathbf{LI}$: sequent calculus for the 1st order intuitionistic logic.
$\mathbf{G^1LI_0}$: sequent calculus $\mathbf{G^1LI}$ restricted to FM_0.
$\mathbf{G^1LI_1}, \mathbf{G^1LI_2}, \mathbf{G^1LI_3}, \ldots$

Theorem

If $\mathbf{PA} \vdash \varphi$ ($\in \Sigma^0_1$), then $\mathbf{G^1LI_0} \vdash \xi \to \varphi$.
Cut elimination for $\mathbf{G^1LI_0}$ implies 1-consistency of $\mathbf{PA}$.
Cut elimination for $\mathbf{G^1LI}_n$ implies 1-consistency of $\mathbf{ID}_n$.

Parameter-free logics and inductive definitions

$\mathbf{LI}$: sequent calculus for the 1st order intuitionistic logic.
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If $\mathbf{PA} \vdash \varphi \ (\in \Sigma^0_1)$, then $\mathbf{G^1LI_0} \vdash \xi \to \varphi$.
Cut elimination for $\mathbf{G^1LI_0}$ implies 1-consistency of $\mathbf{PA}$.
Cut elimination for $\mathbf{G^1LI}_n$ implies 1-consistency of $\mathbf{ID}_n$.

We are now interested in proving cut elimination for $\mathbf{G^1LI_0}$ globally in $\mathbf{ID}_1$ and locally in $\mathbf{PA}$ so that

$$\mathsf{1CON}(\mathbf{PA}) \leftrightarrow \mathsf{CE}(\mathbf{G^1LI_0})$$

is proved in a suitably weak metatheory (eg., $\mathbf{PRA}$).

Ω-rule

私はアルファでありオメガである

Ω-rule: the motivation

Cut elimination for 2nd order logics is tricky, since the reduction step

$$\dfrac{\dfrac{\Gamma \Rightarrow_Y \varphi(Y)}{\Gamma \vdash \forall X.\varphi(X)} \qquad \dfrac{\varphi(\lambda x.\psi) \Rightarrow \Pi}{\forall X.\varphi(X) \Rightarrow \Pi}}{\Gamma \Rightarrow \Pi}\ (CUT)$$

$$\Downarrow$$

$$\dfrac{\Gamma \Rightarrow \varphi(\lambda x.\psi) \qquad \varphi(\lambda x.\psi) \Rightarrow \Pi}{\Gamma \Rightarrow \Pi}\ (CUT)$$

may yield a BIGGER cut formula. Ω-rule (Buchholz 81, Buchholz-Schütte 88, Buchholz 01, Aehlig 04, Akiyoshi-Mints 16, . . .) is a way to resolve this difficulty.

Ω-rule: the idea

The (simplified) Ω-rule for $\mathbf{G}^1\mathbf{LI}_0$:

$$\frac{\{\ \Delta \Rightarrow \Pi^*\ \}_{\Delta \Rightarrow^{\mathbf{LI}}_{Y} \varphi^*(Y)}}{\forall X.\varphi(X) \Rightarrow \Pi}$$

where $*$ is any substitution for 1st order free variables and $\Delta \Rightarrow^{\mathbf{LI}}_{Y} \varphi^*(Y)$ means

- $Y \notin \mathsf{FV}(\Delta)$,
- $\Delta \subseteq \mathsf{Fm}$ (1st order formulas),
- $\mathbf{LI} \vdash \Delta \Rightarrow \varphi^*(Y)$.

"If $\Delta \Rightarrow^{\mathbf{LI}}_{Y} \varphi^*(Y)$ implies $\Delta \Rightarrow \Pi^*$ for any $*$ and $\Delta \subseteq \mathsf{Fm}$, then $\forall X.\varphi(X) \Rightarrow \Pi$."

Ω-rule: the idea

Embedding: We have:

$$\frac{\{\ \Delta \Rightarrow \varphi^*(\lambda x.\psi)\ \}_{\Delta \Rightarrow^{\mathbf{LI}}_{Y} \varphi^*(Y)}}{\forall X.\varphi(X) \Rightarrow \varphi(\lambda x.\psi)}$$

Hence $\forall X$-left can be simulated by Ω.

Collapsing: Consider

$$\frac{\dfrac{\Gamma \Rightarrow_Y \varphi(Y)}{\Gamma \Rightarrow \forall X.\varphi(X)} \qquad \dfrac{\{\ \Delta \Rightarrow \Pi^*\ \}_{\Delta \Rightarrow^{\mathbf{LI}}_{Y} \varphi^*(Y)}}{\forall X.\varphi(X) \Rightarrow \Pi}}{\Gamma \Rightarrow \Pi}\ (CUT)$$

If $\Gamma \Rightarrow^{\mathbf{LI}}_{Y} \varphi(Y)$ holds, then $\Gamma \Rightarrow \Pi$ is one of the premises (with $* = \mathrm{id}$). Hence the (CUT) can be eliminated.

Syntactic cut elimination for $\mathbf{G}^1\mathbf{LI}_0$:

1. Introduce a new proof system based on the Ω-rule by inductive definition.
2. Show that $\mathbf{G}^1\mathbf{LI}_0$ embeds into the new proof system.
3. Apply a syntactic cut elimination procedure.

It works for derivations of 1st order sequents.
(Can be extended to all derivations (Akiyoshi-Mints 16))

Theorem

$\mathbf{ID}_1$ proves that $\mathbf{G}^1\mathbf{LI}_0$ is a conservative extension of $\mathbf{LI}$.

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So the Ω-rule works, but is it logically sound?

Ω-valuation

スライムをゆうしゃのつるぎで倒すのは
大人げないと思う。

Warm-up: conservative extension by MacNeille completion

Let us first give an algebraic proof to

Fact

$\mathbf{G}^1\mathbf{LI}_0$ is a conservative extension of $\mathbf{LI}$.

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(Proof)

Let $\mathbf{L} := \mathsf{Fm}/{\sim}$ be the Lindenbaum algebra for $\mathbf{LI}$.

Let $\overline{\mathbf{L}}$ be the MacNeille completion of $\mathbf{L}$.

Warm-up: conservative extension by MacNeille completion

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The canonical valuation f : Fm $\longrightarrow \mathbf{L}$

$$f(\varphi) := [\varphi]$$

can be extended to $\overline{f}$: FM$_0$ $\longrightarrow \overline{\mathbf{L}}$ since $\overline{\mathbf{L}}$ is complete.

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If $\mathbf{G}^1\mathbf{LI}_0 \vdash \varphi$ with $\varphi \in$ Fm, then $\overline{f}(\varphi) = \top$ by Soundness.

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If $\mathbf{G}^1\mathbf{LI}_0 \vdash \varphi$ with $\varphi \in$ Fm, then $\overline{f}(\varphi) = \top$ by Soundness.
Since $\overline{f} = f$ for Fm, we have $f(\varphi) = [\varphi] = \top$.
That is, $\mathbf{LI} \vdash \varphi$.

MacNeille completion and Ω-rule

Difficulty: the definition of $\overline{f}$ involves

$$\overline{f}(\forall X.\varphi) = \bigwedge_{\xi:\mathsf{Tm}\to\overline{\mathbf{L}}} \overline{f}_{[X\mapsto\xi]}(\varphi)$$

that cannot be formalized in $\mathbf{PA}$.

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Key observation

The Ω-rule is sound w.r.t. $\overline{f} : \mathsf{FM}_0 \longrightarrow \overline{\mathbf{L}}$, though unsound in general.

The reason is that Ω-rule is "similar" to MacNeille.

$$\frac{\{\ \Delta \Rightarrow \Pi^*\ \}_{\Delta \Rightarrow^{\mathbf{LI}}_{Y} \varphi^*(Y)}}{\forall X.\varphi(X) \Rightarrow \Pi} \qquad \frac{\{a \leq y\}_{a \leq x}}{x \leq y}$$

Conservative extension by Ω-valuation

Motivated by this, we introduce the Ω-valuation $f^\Omega : \mathsf{FM}_0 \longrightarrow \overline{\mathbf{L}}$.

$$
\begin{array}{rcl}
f^\Omega(p(\bar{t})) & = & [p(\bar{t})] \\
f^\Omega(t \in X) & = & [t \in X] \\
f^\Omega(\varphi \to \psi) & = & f^\Omega(\varphi) \to f^\Omega(\psi) \\
f^\Omega(\forall x.\varphi(x)) & = & \bigwedge_{t \in \mathsf{Tm}} f^\Omega(\varphi(t)) \\
f^\Omega(\forall X.\varphi(X)) & = & \bigvee\{[\Delta] \in \mathbf{L} : \Delta \Rightarrow^{\mathbf{LI}}_{Y} \varphi(Y) \text{ for some } Y\}
\end{array}
$$

Lemma

$\mathbf{G}^1\mathbf{LI}_0$ is sound w.r.t. the Ω-valuation.

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\end{array}
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Lemma

$\mathbf{G}^1\mathbf{LI}_0$ is sound w.r.t. the Ω-valuation.

Remark: (Altenkirch-Coquand 01) made a similar observation in the context of λ-calculus, but . . .

Local formalization of conservative extension

The argument locally formalizes in $\mathbf{PA}$. Hence:

Theorem (in $\mathbf{PRA}$)

$\mathbf{PA}$ is 1-consistent iff $\mathbf{G}^1\mathbf{LI}_0$ is a conservative extension of $\mathbf{LI}$.

$\mathbf{ID}_n$ is 1-consistent iff $\mathbf{G}^1\mathbf{LI}_n$ is a conservative extension of $\mathbf{LI}$.

Polarity: a uniform framework for MacNeille completion and cut elimination

A polarity is $\mathbf{W} = \langle W, W', R \rangle$ where W, W' are sets and $R \subseteq W \times W'$ (Birkhoff 40).
Given $X \subseteq W$ and $Z \subseteq W'$,

$$\begin{array}{rcl} X^{\rhd} & := & \{z \in W' : \text{ for all } x \in X,\ x \; R \; z\} \\ Z^{\lhd} & := & \{x \in W : \text{ for all } z \in Z,\ x \; R \; z\} \end{array}$$

The pair $(\rhd, \lhd)$ forms a Galois connection:

$$X \subseteq Z^{\lhd} \iff X^{\rhd} \supseteq Z$$

so induces a closure operator on $\wp(W)$:

$$\begin{array}{rcl} \gamma(X) & := & X^{\rhd\lhd} \\ G(\mathbf{W}) & := & \{X \subseteq W : X = \gamma(X)\} \\ X \cup_{\gamma} Y & := & \gamma(X \cup Y) \end{array}$$

Polarity yields MacNeille completion

Lemma

$\mathbf{W}^+ := \langle G(\mathbf{W}), \cap, \cup_\gamma \rangle$ is a complete lattice.
It is a complete Heyting algebra under additional assumptions.

Given a lattice (or Heyting algebra) $\mathbf{A}$,

$$\mathbf{W_A} := \langle A, A, \leq \rangle$$

is a polarity. $X^\triangleright$ is the upper bounds of X and $Z^\triangleleft$ is the lower bounds of Z. Let $\gamma(a) := \{a\}^{\triangleright\triangleleft}$.

Theorem

$\gamma : \mathbf{A} \longrightarrow \mathbf{W}^+_\mathbf{A}$ is the MacNeille completion of $\mathbf{A}$.

MacNeille completion and Dedekind cuts

For example, consider

$$\mathbf{W}_{\mathbb{Q}} := \langle \mathbb{Q}, \mathbb{Q}, \leq \rangle$$

Then for each $X \in G(\mathbf{W})$, $(X, X^{\triangleright})$ is a Dedekind cut. Hence

$$\mathbf{W}_{\mathbb{Q}}^{+} \;\cong\; \mathbb{R} \cup \{\pm\infty\}.$$

Polarity for algebraic cut elimination

We now give an algebraic proof to

Theorem

$\mathbf{G}^1\mathbf{LI}_0$ admits cut elimination.

Define a polarity by

$$
\begin{array}{lcl}
\mathbf{W}_{cf} & := & \langle Seq, Cxt, \Rightarrow^{cf} \rangle \\
Seq & := & \mathsf{FM}_0^* \\
Cxt & := & \mathsf{FM}_0^* \times (\mathsf{FM}_0 \cup \{\emptyset\}) \\
\Gamma \Rightarrow^{cf} (\Sigma, \Pi) & \Leftrightarrow & \Gamma, \Sigma \Rightarrow \Pi \text{ is cut-free provable in } \mathbf{G}^1\mathbf{LI}_0.
\end{array}
$$

Fact

$\mathbf{W}_{cf}^+$ is a complete Heyting algebra such that

$$\Gamma \in \varphi^{\lhd} \iff \Gamma \Rightarrow^{cf} \varphi.$$

Ω-valuation again

One could use the "reducibility candidates" technique as in (Maehara 91) and (Okada 96), but it is too strong for $\mathbf{G}^1\mathbf{LI}_0$. It doesn't (locally) formalize in $\mathbf{PA}$.

Ω-valuation f : $\mathsf{FM}_0 \longrightarrow \mathbf{W}^{+}_{cf}$

$$
\begin{array}{lcl}
f^\Omega(p(\bar{t})) & = & p(\bar{t})^{\lhd} \\
f^\Omega(t \in X) & = & (t \in X)^{\lhd} \\
f^\Omega(\varphi \to \psi) & = & f^\Omega(\varphi) \to f^\Omega(\psi) \\
f^\Omega(\forall x.\varphi(x)) & = & \bigcap_{t \in \mathsf{Tm}} f^\Omega(\varphi(t)) \\
f^\Omega(\forall X.\varphi(X)) & = & \forall X.\varphi(X)^{\lhd} \\
 & = & \{\Delta \in Seq : \Delta \Rightarrow^{cf}_{Y} \varphi(Y) \text{ for some } Y\}^{\rhd\lhd}
\end{array}
$$

Lemma

$\mathbf{G}^1\mathbf{LI}_0 \vdash \Gamma \Rightarrow \Pi$ implies $f^\Omega(\Gamma) \subseteq f^\Omega(\Pi)$ (Soundness).
$\varphi \in f^\Omega(\varphi) \subseteq \varphi^\lhd$ for any $\varphi \in \mathsf{FM}_0$ (Completeness).

Now cut elimination for $\mathbf{G}^1\mathbf{LI}_0$ follows easily.

(Proof) Suppose $\mathbf{G}^1\mathbf{LI}_0 \vdash \varphi \Rightarrow \psi$.
Then $f^\Omega(\varphi) \subseteq f^\Omega(\psi)$ by Soundness.
$\varphi \in f^\Omega(\varphi) \subseteq f^\Omega(\psi) \subseteq \psi^\lhd$ by Completeness.
So $\varphi \Rightarrow \psi$ is cut-free provable.

We have shown provability = cut-free provability.

So a fortiori we obtain:

Theorem

$\mathbf{W}^{+}_{cf}$ is the MacNeille completion of the Lindenbaum algebra for $\mathbf{G}^1\mathbf{LI}_0$.

algebraic c.elim = MacNeille compl. + Ω-valuation

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Theorem

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algebraic c.elim = MacNeille compl. + Ω-valuation

Theorem (in $\mathbf{PRA}$)

$$\begin{aligned} \mathsf{1CON}(\mathbf{PA}) &\leftrightarrow \mathsf{CE}(\mathbf{G}^1\mathbf{LI}_0) \\ \mathsf{1CON}(\mathbf{ID}_n) &\leftrightarrow \mathsf{CE}(\mathbf{G}^1\mathbf{LI}_n) \end{aligned}$$

For the lambda calculus audience

今北産業

Why does metatheory matter?

We have been careful in which metatheory the theorem is proved.

Does it matter if one is only interested in the TRUTH?

Yes! Since a proper metatheory consideration often leads to an interesting TRUTH such as

iterated System $\mathbf{T}$ = parameter-free System $\mathbf{F}$.

Parameter-free System $\mathbb{F}$

Type$_0$ is defined by:

$$A, B \ ::= \ X \mid A \Rightarrow B \mid \forall X.C,$$

where C is a simple type s.t. Fv$(C) \subseteq \{X\}$.

$\mathbf{F}^p_0 :=$ System $\mathbf{F}$ with types restricted to Type$_0$.

Eg. $\mathbf{N} := \forall X.(X \Rightarrow X) \Rightarrow (X \Rightarrow X) \quad \in$ Type$_0$.

Clearly Rep$(\mathbf{T}) \subseteq$ Rep$(\mathbf{F}^p_0)$.

How do you prove the converse?

Theorem (Akiyoshi-T. 16)

$\mathbf{ID}_1 \vdash \mathsf{SN}(\mathbf{F}_0^p)$.
$\mathbf{PA} \vdash \Phi\text{-}\mathsf{SN}(\mathbf{F}_0^p)$ for any finite $\Phi \subseteq \mathsf{Type}_0$.

The 2nd statement implies: for every closed term $M : \boldsymbol{N} \Rightarrow \boldsymbol{N}$ of $\mathbf{F}_0^p$,

$$\mathbf{PA} \vdash \forall x \exists y.\ \text{“}M\underline{x} =_\beta \underline{y}\text{''},$$

hence $\mathsf{Rep}(\mathbf{F}_0^p) \subseteq \mathsf{Total}(\mathbf{PA}) = \mathsf{Rep}(\mathbf{T})$.

Theorem (Altenkirch-Coquand 01)

$\mathsf{Rep}(\mathbf{F}_0^p) = \mathsf{Rep}(\mathbf{T})$.

iterated System $\mathbf{T}$ = parameter-free System $\mathbf{F}$.

For the nonclassical logics audience

Beyond classical and intuitionistic: substructural logics

Recall:

Theorem (Harding-Bezhanishvili 04)

$\mathcal{HA}$ and $\mathcal{BA}$ are the only nontrivial subvarieties of $\mathcal{HA}$ closed under MacNeille completions.

On the other hand, one finds abundant of examples in substructural logics and associated residuated lattices.

Theorem (Ciabattoni-Galatos-T. 12)

- There are infinitely many varieties of residuated lattices closed under MacNeille completions.
- So there are infinitely many substructural logics that admit algebraic cut elimination.

For intermediate logics, a useful framework is hypersequent calculus. Associated completion is hyper-MacNeille completion.

Theorem (Ciabattoni-Galatos-T. 08, 17)

- There are infinitely many subvarieties of $\mathcal{HA}$ closed under hyper-MacNeille completions.
- So there are infinitely many intermediate logics that admit algebraic cut elimination in hypersequent calculi.

On the other hand, there are also counterexamples for cut elimination/completion in substructural logics. That is WHY substructural logics are interesting!

Theorem

- There is an MV algebra (Chang's chain) which cannot be embedded into a complete MV algebra.
- That is, $\mathcal{MV}$ is not closed under any completion

- Hence Łukasiewicz infinite-valued logic cannot be conservatively extended with infinitary $\bigwedge$.
- That is, Ł has NO "good" proof system (although some exist . . .).

Conclusion

- Ω-rule is valid for the MacNeille completion of the Lindenbaum algebra.
- This leads to algebraic cut elimination for $\mathbf{G}^1\mathbf{LI}_0$ based on MacNeille completion + Ω-valuation.

Target	Fragments	Full higher-order logics
Algebraic	MacNeille + Ω-valuation	MacNeille + reducibility candidates
Syntactic	Ω-rule	Takeuti's Conjecture

1-consistency of $\mathbf{ID}_n$ $=$ cut-elimination for $\mathbf{G}^1\mathbf{LI}_n$
iterated System $\mathbf{T}$ $=$ parameter-free System $\mathbf{F}$.

Final Word

人生面白いほうに三千点。さらに倍。
（羽海野チカ『ハチミツとクローバー』
より）